

Multiple Perspectives on the Important Concepts for Understanding Recursion and Induction

The study of recursion and mathematical induction present a number of challenging and illusive concepts. Through a familiar classroom example, this brief addresses these concepts and provides some answers to the following questions:

What is recursion?

What are recursive processes?

What is induction?

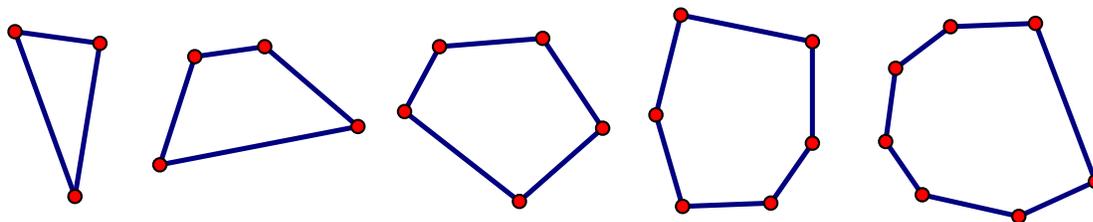
What are inductive processes?

What is proof by induction?

Vignette #1: An informal conversation in Denmark

Mogens, a grandfather in Denmark, sits with his eight-year-old granddaughter, Marie, doing mathematics. He draws the polygons in Figure 1.

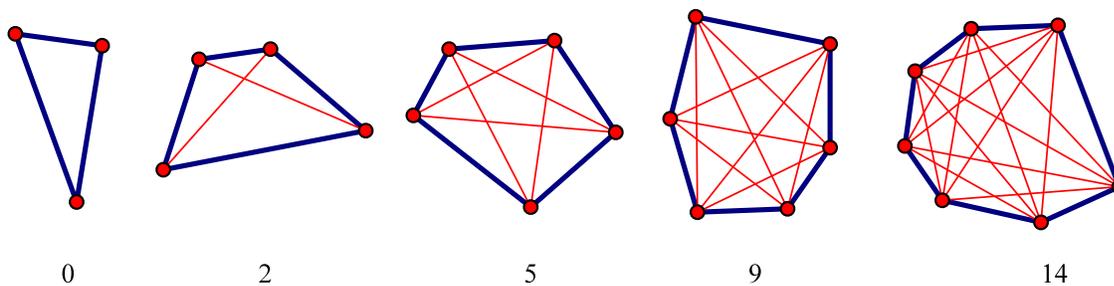
Figure 1: Five convex polygons



He asks, “How many diagonals can you draw in each shape?”

Together they draw the diagonals in Figure 2.

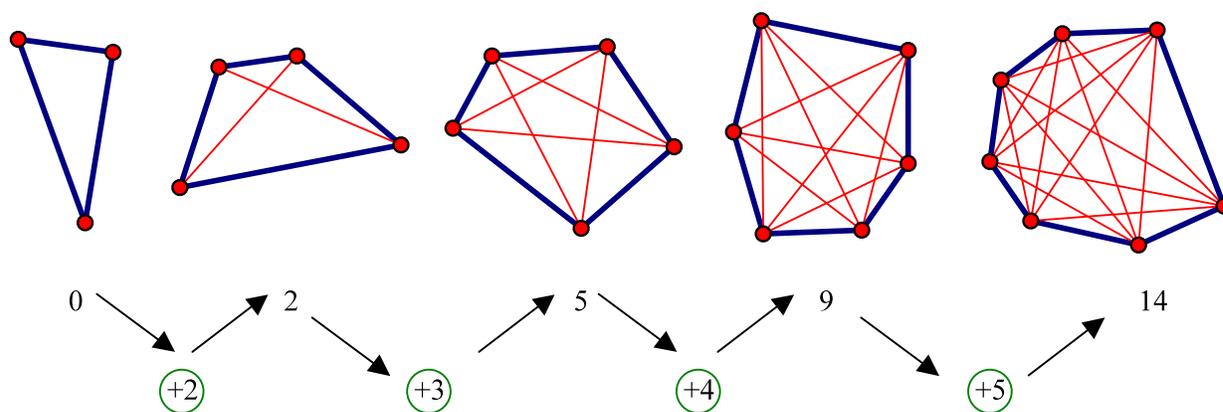
Figure 2: Five convex polygons with their diagonals



Mogens asks, “How many diagonals would be in an 8-gon? How about a 9-gon?”

Marie notices the pattern of addition, seen in Figure 3, when going from one polygon to the next, and says that there will be six more, or 20 diagonals, in an 8-gon, and seven more than that, or 27 diagonals, in a 9-gon.

Figure 3: Number pattern associated with diagonals in polygons



Mogens then asks, “How many diagonals do you think would be in a polygon with 20 sides?”

Marie calculates: $14 + 6 + 7 + \dots + 18$ to determine the number of additional diagonals in the 20-gon. She reasons that there are 170 diagonals.

Commentary on Vignette #1:

In the interaction, Mogens prompts Marie to engage in **inductive thinking**. By this, we mean that Marie looks at data (the number of diagonals in each polygon) and makes a conjecture about how the pattern continues. Marie’s drawings and inductive analysis lead her to notice numerical recursion. Specifically, the difference in numbers of diagonals between two consecutive polygons is one greater than the previous difference. So, Marie observes an inductive pattern in the numbers of diagonals.

Vignette #2: A classroom in Peru

In Peru, Dr. Villavicencio says to her class, “Yesterday, you noticed some patterns in the diagonal problem. Today, we will work on a way to determine the number of diagonals in a 20-gon without having to calculate the number of diagonals in each of the previous polygons.”

The students work in groups, and one group eventually shares Table 1 where “ D_n ” denotes the number of diagonals of a polygon with n sides. Students recognized one of the patterns as the sum of consecutive integers.

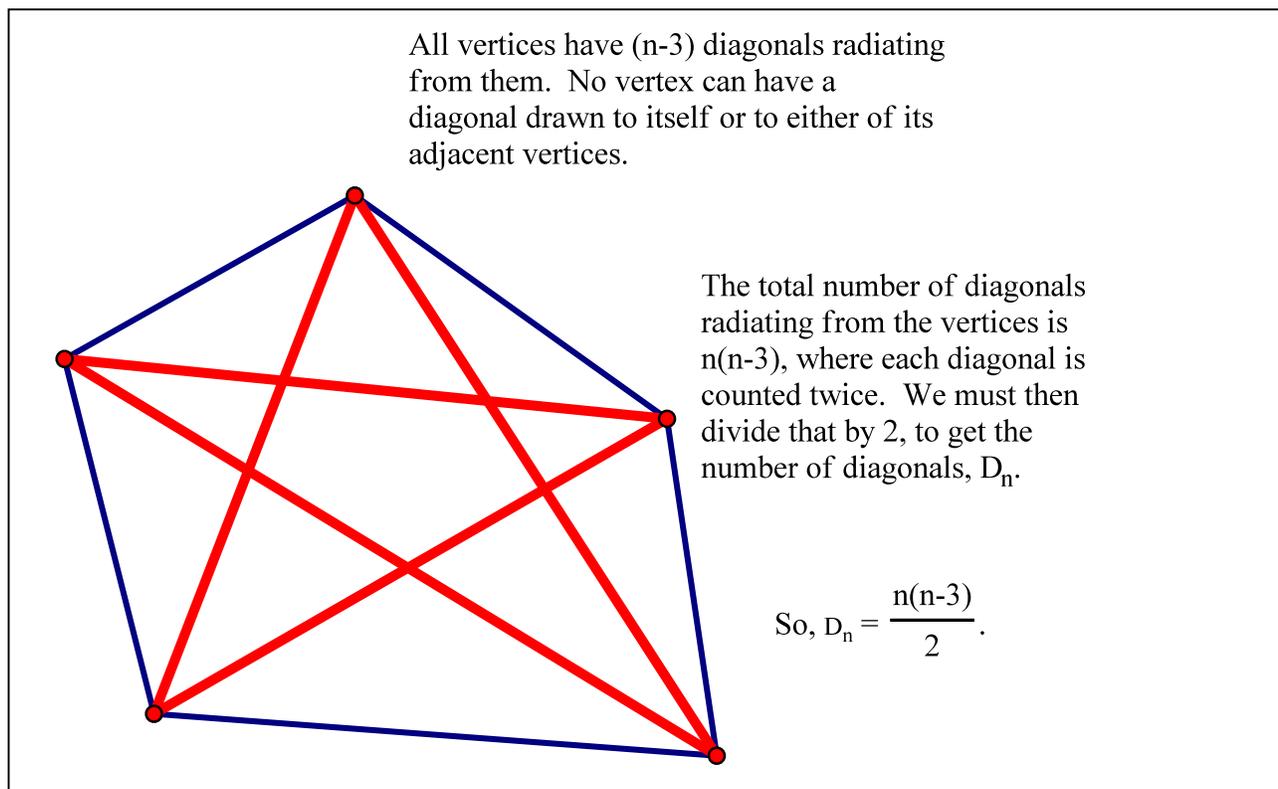
Table 1: Patterns seen in the number of diagonals in a polygon problem

n	D_n	D_n showing all calculations	D_n as a sum of consecutive integers
3	0	0	0
4	2	2	2
5	5	$2 + 3$	$D_5 = \frac{(2+3)2}{2} = 5$
6	9	$2 + 3 + 4$	$D_6 = \frac{(2+4)3}{2} = 9$
7	14	$2 + 3 + 4 + 5$	$D_7 = \frac{(2+5)4}{2} = 14$
...
20	?	$2 + 3 + 4 + 5 + \dots + 18$	$D_{20} = \frac{(2+18)17}{2} = 170$
...
100	?	$2 + 3 + 4 + 5 + \dots + 98$	$D_{100} = \frac{(2+98)97}{2} = 4850$
...
n	?	$2 + 3 + 4 + 5 + \dots + (n - 2)$	$D_n = \frac{(2+(n-2))(n-3)}{2}$ $D_n = \frac{n(n-3)}{2}$

Dr. Villavicencio asks, “Can we explain the formula for any n -gon using the diagrams of the polygons?”

The students work in groups, and one presents the discussion in Figure 4.

Figure 4: Student observation about the number of diagonals in an n -gon



Commentary on Vignette #2

In Table 1, the students made some inductive observations: one, that the last number in the sum in column three is $(n - 2)$; two, that the number of terms in the series in column three is $(n - 3)$. They also made use of deductive reasoning, when they identified that the sum of consecutive numbers can be calculated using $S_n = \frac{n(a+tn)}{2}$, a formula which they knew to be appropriate for this situation. This approach yielded a non-recursive and explicit general term.

Then, Dr. Villavicencio shifts her students' attention away from inductive pattern-seeking by asking them to use the diagram to explain their explicit formula. For their explanation, students rely on their experiences drawing and counting the diagonals. We are calling this a "proof by explanation." This task highlights a connection between algebra and a visual pattern.

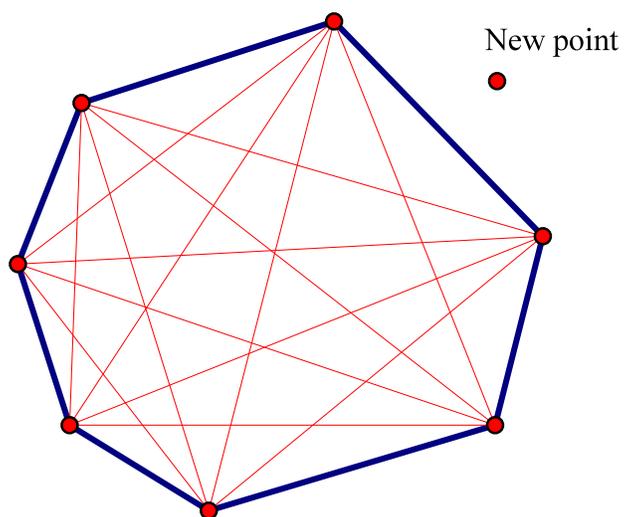
Vignette #3: A video conference between Canada and the United States

In a video link between a classroom in Canada and one in the United States, students learn to view the problem above recursively. The teachers begin by asking how different ways to draw an image can illuminate different ways of seeing a pattern, "Today, we return to the diagonal

problem and see what we can learn if we draw the images recursively with a successive image arising from its predecessor. Here is a 7-gon, with all diagonals drawn.”

The screens in both classrooms show Figure 5.

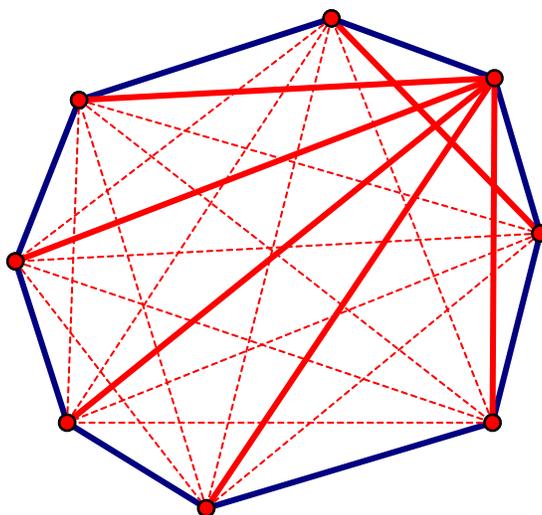
Figure 5: A 7-gon with its diagonals



A voice-over says, “Watch how a new point can be used to transform the 7-gon to an 8-gon.” The teachers manipulate the figure above until the image in Figure 6 is shown.

Figure 6: An 8-gone created from a 7-gon

The bold segments are the new diagonals formed when the new vertex was added.



After a discussion of how this method of drawing an image is different than previous methods, students acknowledge that this recursive lens on the problem offers an additional and distinct understanding. They work groups to make sense of the following written text:

Assume we have an n -gon with D_n diagonals. We want to establish the number, D_{n+1} , of diagonals in an $(n+1)$ -gon by determining how to obtain D_{n+1} from D_n .

The $(n+1)$ -gon results when you add a new vertex to the n -gon. Adding this vertex introduces $(n-2)$ new diagonals from the non-neighboring vertices plus one new diagonal connecting the two neighbors of the new vertex. This new diagonal in the $(n+1)$ -gon was formerly a side of the n -gon.

So, the number of diagonals in the $(n+1)$ -gon is $D_{(n+1)} = D_n + (n-2) + 1 = D_n + (n-1)$

Therefore, we have established D_{n+1} recursively in terms of something “done” to D_n , namely adding $(n-1)$ diagonals to D_n .

Carrying this process further backward, we get $D_{n+1} = D_n + (n-1) = D_{n-1} + (n-2) + (n-1) = 2 + \dots + (n-2) + (n-1)$.

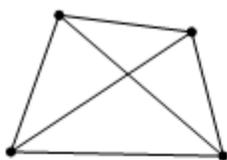
Commentary on Vignette #3

In this video-conferencing experience, students’ attention shifts from “numerical recursion” to “geometric recursion.” That shift provides yet another way of justifying the addition of consecutive integers. The students make use of recursive notation, expressing a term as a function of the previous term.

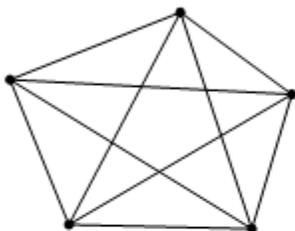
Vignette #4: A classroom in Cambodia

In a classroom in Cambodia, the teacher, Si Ath, continues the discussion of the previous day’s activity about diagonals in a polygon, when the students counted the number of diagonals. He refers to Figure 7 from yesterday’s work.

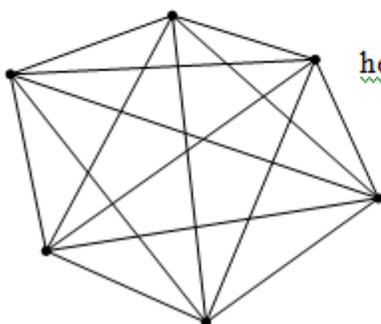
Figure 7: Previous day’s work on polygons from a Cambodian class



quadrilateral : $D_4 = \frac{4 \times 3}{2} - 4 = \frac{4}{2}(3-2) = \frac{4}{2}(4-3) = 2$



pentagon : $D_5 = \frac{5 \times 4}{2} - 5 = \frac{5}{2}(4-2) = \frac{5}{2}(5-3) = 5$



hexagon : $D_6 = \frac{6 \times 5}{2} - 6 = \frac{6}{2}(5-2) = \frac{6}{2}(6-3) = 9$

Si Ath wishes to extend the previous exploration saying, “Yesterday, we used inductive reasoning to conclude that $D_n = \frac{n(n-3)}{2}$. How certain can we be that this is true?”

After some discussion, Si Ath asks his students to show that $D_n = \frac{n(n-3)}{2}$ using proof by mathematical induction. He collaborates with them to come up with the proof. One group’s proof is as follows:

Proof by induction:

Base case (a quadrilateral): $D_4 = \frac{4(4-3)}{2} = 2$ is correct

Assume that for some k -gon, $D_k = \frac{k(k-3)}{2}$ is true.

We must prove that we can use the assumption for the k -gon and extend that result for a $(k+1)$ -gon where D_{k+1} is the number of the diagonals of a $(k+1)$ -gon. This approach uses the recursive idea from Vignette 3 where a new point was added to the k -gon to obtain a $(k+1)$ -gon. When the point is added, there will be $(k-2)$ diagonals attached to the new

point and one diagonal attached to the two neighboring vertices resulting in the following calculation:

$$\begin{aligned}
 D_k + (k - 2) + 1 &= \frac{k(k - 3)}{2} + (k - 2) + 1 \\
 &= \frac{k(k - 3) + 2(k - 2) + 2 \cdot 1}{2} \\
 &= \frac{k(k - 3) + 2(k - 2 + 1)}{2} \\
 &= \frac{k^2 - 3k + 2k - 2}{2} = \frac{k^2 - k - 2}{2} \\
 &= \frac{(k + 1)(k - 2)}{2} \\
 &= \frac{(k + 1)[(k + 1) - 3]}{2} \\
 &= D_{k+1}
 \end{aligned}$$

Thus, D_{k+1} is true when D_k is true.

Therefore, the formula for the number of diagonals in an n -gon is true when n is greater than or equal to 4.

Commentary on Vignette #4

Si Ath's lesson extends the students' thinking. They move from the organizing and analyzing of data, to an inductively conducted general term, and then to a formal proof by mathematical induction.

Concluding Commentary

These examples in the vignettes represent four different approaches to solving the same problem: Determine the number of diagonals in an n -gon (a polygon with n sides/vertices).

In the first vignette with a conversation between Mogens and his granddaughter, five consecutive concrete cases were explored and the number of diagonals counted and recorded. They observed that the number of additional diagonals needed to go from a polygon of 3 sides to 4 sides, from 4 sides to 5 sides and so on, increased by one at every step.

From this, they formed the hypothesis that this pattern of growth would continue: in every step from an n -gon to a $(n+1)$ -gon, the number of new diagonals increased by one. The sequence of diagonals, 0, 2, 5, 8, ... gives rise to the number sequence from the first set of differences: 2, 3, 4, ..., 18, This sequence is the increase in new diagonals as you move from term to term from which it was suggested that a 20-gon has 170 diagonals. This approach illuminates what happens

in each step, but not why it happens. The outcome, 170 diagonals, can be perceived as a result of informed and systematic guessing from the pattern observed.

Vignette #2 depicts a continuation of Vignette # 1 in two ways: (1) it keeps track of how the additional number of diagonals in the transition from n to $(n+1)$ grows as n grows [The last number added to obtain D_{n+1} is $(n+1) - 2 = n - 1$]; (2) the formula for the sum of consecutive integers can be used to achieve a general formula for D_n . This vignette also depicts a generalization based on induction from the observed pattern, but the result obtained appears to be general for any n . There is still no element of “why,” only of “what.”

In Vignette #2, when the students are asked to reference a diagram to explain why, they use a third approach. This one is a direct approach, based on an analysis of the definition of a diagonal in an arbitrary n -gon and on a resulting combinatorial argument relying on enumeration of the diagonals emerging from each vertex and keeping track of double counts. There is no recursive or inductive process involved in this approach, which illuminates the geometrical combinatorial mechanisms involved in the problem.

In Vignette #3, the focus is on how an $(n+1)$ -gon can be perceived as resulting from an n -gon to which something “is being done”: adding a new vertex. This is an example of recursive thinking where the $(n+1)$ th stage is construed in terms of the n th stage. In this approach, the mechanism behind the transition from the n th to the $(n+1)$ th stage is transparent.

All approaches to the initial problem result in the same formula for D_n , based on reasoning conveying varying forms and degrees of insight. To ensure the conclusion is valid, Vignette #4 illustrates proof by mathematical induction, based on the induction step, $D_{n+1} = D_n + (n - 1)$ and the initial observation $D_4 = \frac{4(4-3)}{2}$.

The different approaches, based on different sorts of inductive and recursive reasoning, provide insights into the solution of the problem – hence, into different facets of the problem. The collection of approaches enlarges our insights.

Terms and their meaning

Based on the vignettes, we propose definitions of key concepts related to recursion and induction.

Both recursion and induction are related to transition between discrete stages that can be indexed by natural numbers. So we have a (potentially) infinite sequence of stages starting with stage S_1 :

$S_1, S_2, \dots, S_{n-1}, S_n, S_{n+1}, \dots$

Consider stage S_n . If the focus is on how to move to subsequent stages, S_{n+1}, S_{n+2}, \dots , we deal with an **inductive process**:

$$S_n \rightarrow S_{n+1} \rightarrow \dots$$

If, instead, the focus is on how to get to stage S_n from stage S_{n-1} and backward from there, we are dealing with a **recursive process**:

$$S_1 \dots \leftarrow S_{n-1} \leftarrow S_n.$$

From the perspective of transition through stage S_n , inductive and recursive processes can be perceived as two sides of the same coin. Looking back at how we got from S_{n-1} to S_n , we realize that this involved moving forward from S_{n-1} to S_n .

$$S_{n-1} \leftrightarrow S_n$$

However, even though inductive and recursive processes then display an intrinsic duality between them, there is an important difference as well. A recursive process stops when it arrives back at stage S_1 , whereas an inductive process (in principle) can be continued forever.

Inductive and recursive processes include, but are not confined to, dealing with mathematical objects, phenomena, processes or situations. In relation to mathematics, however, we have particular forms of inductive and recursive processes called **recursion** and **induction**.

More specifically, in mathematics recursion occurs when a sequence of mathematical objects, O_n , has the property that O_1 is given and for every n , O_n is specified entirely in terms of some or all objects O_k , where $k < n$. Looking at mathematical objects in this manner through “recursion lenses” can be described as **recursive thinking**.

Induction occurs in mathematics when properties of a sequence of mathematical objects are derived by way of generating properties of O_{n+1} from certain properties of O_n . Looking at properties of mathematical objects in this manner through “induction lenses” can be described as **inductive thinking**.

A completely generalizable form of inductive thinking is found in the so-called **proof by mathematical induction**. Despite its name, proof by mathematical induction is actually a form of rigorous mathematical deduction. In mathematical induction, we prove that every member of a sequence of mathematical objects, O_n , has a certain property P by establishing two assertions:

- a) O_1 has the property P (i.e., $P(1)$ holds);
- b) **If** for any given k , O_k has the property P (i.e., $P(k)$ holds), **then** O_{k+1} also has the property P (i.e., $P(k+1)$ holds).

In part b, there is an implication: For every k , $P(k)$ (implies) $P(k+1)$.

Proof by mathematical induction relies on the principle of induction, a core property of the set of natural numbers:

Suppose P is a predicate statement in natural numbers. Let M be a subset of natural numbers k in N (the set of all natural numbers) satisfying P (i.e., k belongs to M if and only if $P(k)$ holds). If M also satisfies:

- a) 1 belongs to M ;
- b) For every k in N , if k is in M , then $k+1$ is in M as well,

then $M = N$.

In other words, every n in N satisfies P (i.e., $P(n)$ holds for every n). This principle is derived from the mathematical fact that the set of natural numbers for which P is **not** satisfied must have a first element.