

## The Power of Recursion and Induction

This brief focuses on three domains to elaborate on the question of why it is important to teach recursion and mathematical induction: mathematics, mathematical modeling, and technology.

### **Mathematics**

Some concepts and processes lend themselves to being described using recursion or induction. Recursion is everywhere in mathematics. For example, the set of natural numbers is defined recursively. Some functions are defined recursively, and recursion appears in natural phenomena. For example, the Fibonacci numbers, defined by  $F(1) = 1$  and  $F(2) = 1$ ; and  $F(n) = F(n - 1) + F(n - 2)$ , appear in biological settings. Recursion in the school curriculum is used to teach such topics as mathematical induction and sequences. In some countries recursion has a substantial presence in the school curriculum; in others it is marginal. Contemporary technological tools make it easy to use recursion. Moreover, they enable the recursive process to become visible as described below.

Because many topics in the mathematics curriculum lend themselves to recursive thinking, it recursive thinking as a context may be used to show coherence in the development of these topics. For example, sequences, series, and tessellations can be developed using recursive thinking to show how subsequent values are obtained from previous values. Financial markets and formulae for annuities, mortgages, present values, and future values lend themselves to recursion. Consequently, these topics may be taught meaningfully by using recursive-thinking processes. It is worth noting that relationships derived by generating subsequent values from previous values can be verified and proved by the process of mathematical induction to ensure that the relationships give reliable subsequent values, once the initial values are known.

Recursion and induction play an important role in concepts and processes from other areas of mathematics. For example, the Newton-Raphson algorithm used to find solutions in analytical contexts is defined recursively. Iterative processes can produce results successively closer to a desired result and often yield a better understanding of the problem and possible solutions (see example 8 in the appendix).

### **Modeling**

Dealing with recursion enhances a focus on both modeling and relational thinking. Teaching with recursion often has students finding a recursive formula for a given problem, e.g., number of diagonals in  $n$ -polygons. Teaching this way, the process actually models the situation. The modeling process in general involves several stages (see, for example, *Mathematical modelling and application in secondary education. The 14<sup>th</sup> ICME study*. (2007)), but in particular one of the stages, *mathematization*, is often hard to do. This is the process where the students introduce the language of mathematics in order to solve the problem. When students experiment and struggle to describe a recursive situation using the formal language of mathematics, they are working with the task of

mathematization. This process rests on a comprehensive understanding of the problem and therefore focuses on the cognitional scheme present in students' minds. In other words, mathematization is a process that stimulates relational understanding.

Another important aspect of mathematization is when the mathematization itself is conceived as a means. Namely that, when trying to identify and describe the main features of an exploited pattern, focus is drawn to a deep understanding of what is going on and how it relates to the cognitional scheme already present in students' minds. In other words, mathematization is a process that stimulates relational understanding.

### **Technology**

Recursion is related to technology in two ways: in programming and in the way technology is used. Within the technology itself, some problems can be solved by

programming. For example, in calculating  $1 + \frac{x}{1} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$ , the approximate

value of  $e^x = \exp(x)$  where  $x$  is real and  $n$  is nonnegative integer (see the appendix examples for details).

Many applications of technology involve spreadsheets. Spreadsheets make it possible in an easy way to produce tables based on recursive formulas. For example, consider the population ( $P_n$ ) in year  $n$  as a function of the population ( $P_{n-1}$ ) in year  $n - 1$  plus the number (with rate  $b$  times the population  $P_{n-1}$ ) of newborns minus the number of deaths (with rate  $d$  times the population  $P_{n-1}$ ) plus the net immigration ( $I$ ) in absolute numbers. Analytically, this could be a rather complicated function, but described as a recursive process, it is straightforward and is given as:

$$P_n = P_{n-1} + bP_{n-1} - dP_{n-1} + I = P_{n-1}(1 + b - d) + I$$

In the spreadsheet shown in Table 1, the initial values are A2:=1 and B2:=5,500,000. The rest of the numbers in the column A are produced via  $f_n = f_{n-1} + 1$ , so for example A3=A2+1. Table 1 illustrates how the numbers are calculated.

Table 1: Calculation of Population

	A	B	C	D	E
1	Year	Population #			
2	1	5.500.000,00			
3	2	5.500.500,00	Pn=Pn-1-Dn+Bn+In		
4	3	5.500.992,50	Pn=Pn-1-d*Pn-1+bPn-1+l		
5	4	5.501.477,61	Pn=(1-d+b)Pn-1+l		
6	5	5.501.955,45			
7	6	5.502.426,12			
8	7	5.502.889,72	birthrate	0,015	
9	8	5.503.346,38	deathrate	0,03	
10	9	5.503.796,18	Net. Immigration	83000	
11	10	5.504.239,24			
12	11	5.504.675,65			
13	12	5.505.105,52			
14	13	5.505.528,93			
15	14	5.505.946,00			
16	15	5.506.356,81			
17	16	=B16*(1+\$E\$8-\$E\$9)+\$E\$10			
18					
19					
20					

Thus, recursion and iteration have value in real world contexts while other examples are primarily in a mathematical context. An interesting question for mathematics educators is, would the competence and mathematics of spreadsheet users be enhanced through training *explicitly* in the concept and notation of recursion. If the answer is yes and it is agreed that the use of spreadsheet is becoming a common thing all over the world, there is a very strong argument for the teaching of recursive functions.

### Final remarks

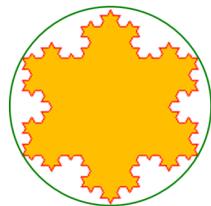
When recursion appears in the secondary school curriculum, it is often included in teaching the topics of mathematical induction and sequences. Contemporary technological tools make it easy to introduce recursion in schools. Moreover, such tools enable the recursion process to become even more visible to students. Yet, often the purpose for including recursion and mathematical induction in the school curriculum is neither well articulated nor clear. Therefore, the place of recursion in the secondary school curriculum needs to be elaborated and made explicit so that decision makers, curriculum developers, and teachers can make informed choices about what to teach and why. In this brief to show the importance of including recursion in the school curriculum, the focus is on mathematics, modeling and technology. Recursion can be a useful approach for solving problems in cases when we can transform a complex problem using successive steps to a simpler problem, which we know how to solve, by iterating the same process. Together with mathematical induction, recursion allows us to move from a complex to a simple problem and then back to solving the complex problem. Still, care is needed to ensure that recursion and mathematical induction are used productively in the school curriculum, consistent with available facilities in different contexts. The appendix includes examples from Korean textbooks and the use of technology where recursion plays a major role. All of the examples could be useful for thinking in this direction.

### Reference

International Commission on Mathematics Instruction. (2007). *Modelling and applications in secondary education, The 14<sup>th</sup> ICMI study*. Blum, W., Galbraith, P. L., Henn, H., and Niss, M. (eds.). (2007). New York: Springer –Science Business Media, LLC.

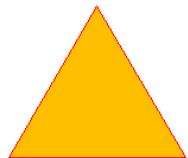
## Appendix

1. What is the area of the following shape (the Koch snowflake) where  $r$  is the diameter of the circle? What is its perimeter?



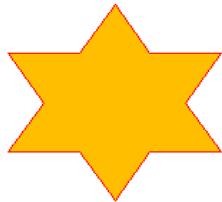
### **Analysis of the Problem:**

This is a difficult problem to solve, but the following is a much easier problem. What is the area of the equilateral triangle shown where  $s$  is the length of a side? What is its perimeter?



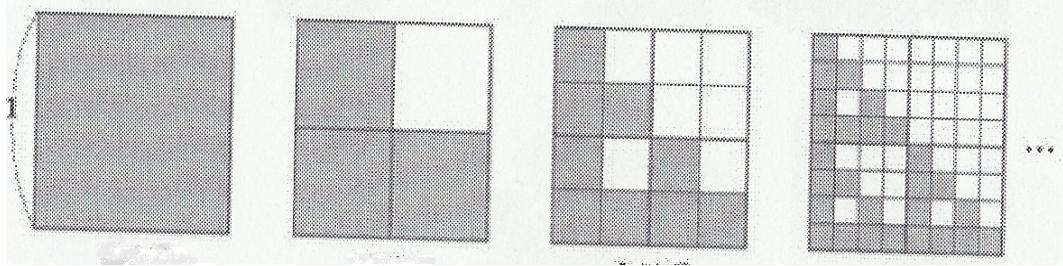
How can solving the simple triangle problem help in solving the complex one?

If you remove the middle one-third of each side of the equilateral triangle and replace it with two segments congruent to the segment you removed you will get the shape below. Can you find its area and perimeter?



If you continue this inductive process you will produce the Koch snowflake in the complex problem after a finite number of steps making it clear how to calculate its area and perimeter.

2. The area of colored squares: From the square, first remove 1/4 of the square as shown in the second figure and repeat the process as shown. Write a recursive formula for the sequence of the areas of colored squares.

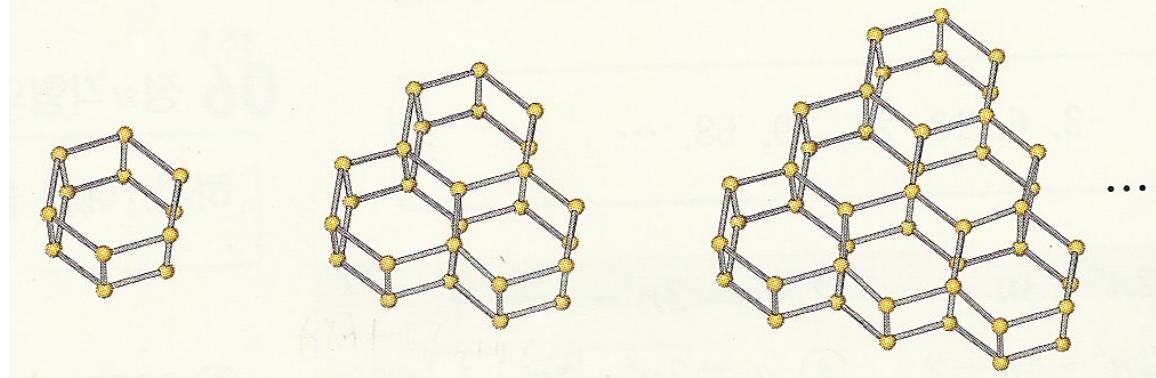

**Analysis of the problem:**

If we count the squares, we obtain the following:

Step	0	1	2	3	...	$n$
Number of colored squares	1	3	9	27	...	$3^n$
Colored area	1	$3/4$	$9/16$	$27/64$	...	$3^n/4^n$

Using recursion makes the problem somewhat simpler than other methods. The number of colored squares is 1, 3, 9, 27, 81, ..., and a recursive formula for this number of squares is  $a_1 = 1$ ;  $a_{n+1} = 3a_n$ . If the area of the original square is 1, then we can write a recursive formula for the sequence of areas, 1,  $3/4$ ,  $9/16$ ,  $27/64$ , ... as follows:  $a_1 = 1$ ;  $a_{n+1} = (\frac{3}{4})a_n$ .

3. By using sticks, make a hexagon-pole (6-pole) as shown below on the left. Use the hexagon-pole (6-pole) to build more complex conglomerates as shown below. What is the number of sticks needed to build the 20<sup>th</sup> conglomerate? And what is the number of sticks needed to build the  $n^{\text{th}}$  conglomerate?


**Analysis of the problem:**

If we count the number of sticks, we have the following:

Conglomerate	1	2	3	4
Number of sticks	18	43	76	117

Thinking recursively, we have the following:

$$a_2 = a_1 + 25$$

$$a_3 = a_2 + 33$$

$$a_4 = a_3 + 41$$

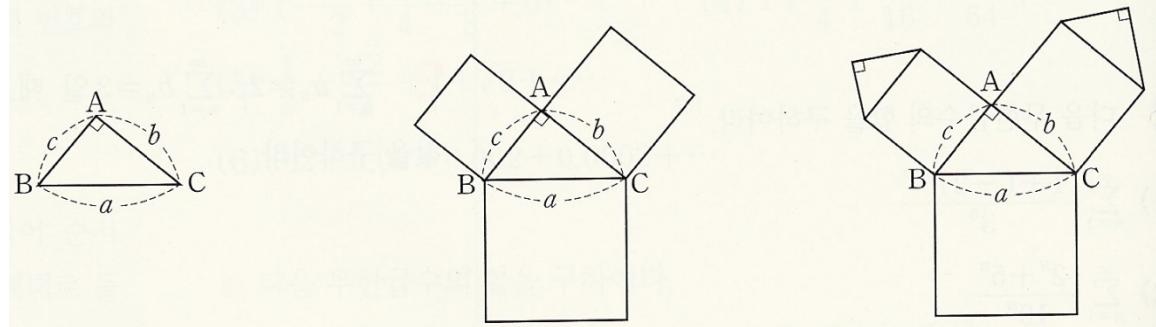
...

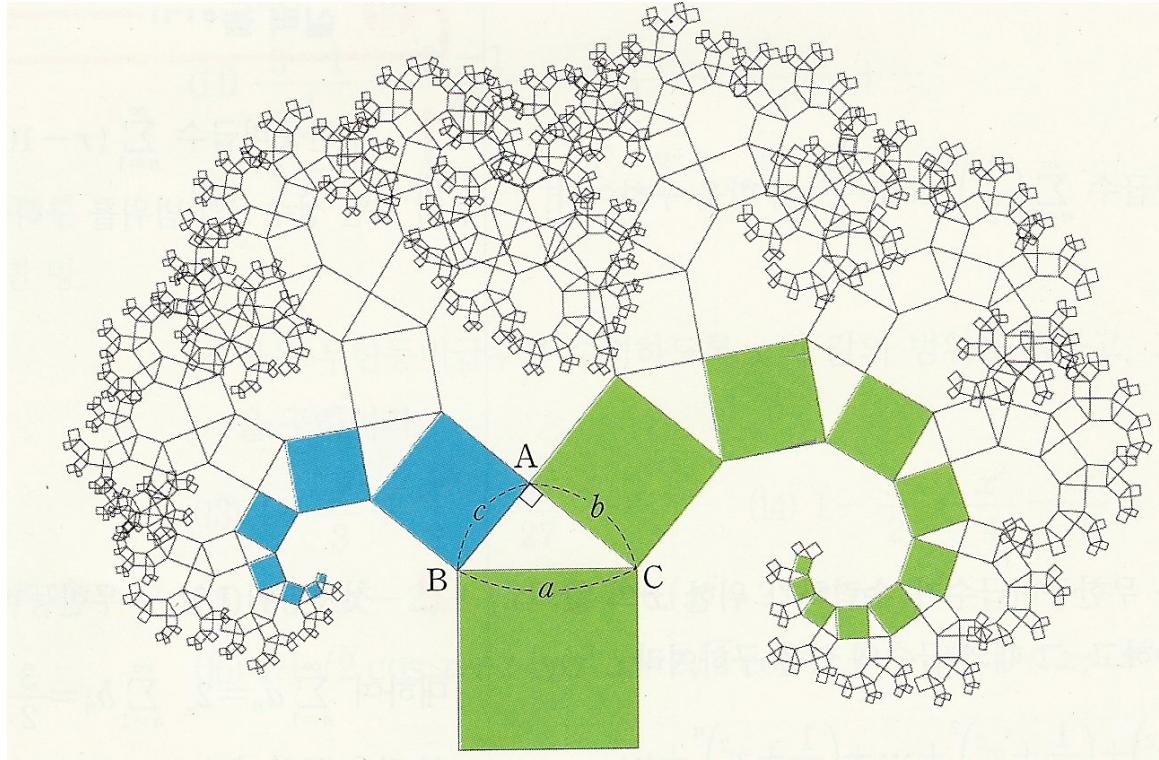
The terms appear to increase by 8. If this is true, the general term is  $25 + 8(n - 1) = 8n + 17$ . [The proof that this is the general term and holds in all cases requires a proof by mathematical induction.]

Thus, the recursion formula would be  $a_1 = 18$ ;  $a_{n+1} = a_n + 8n + 17$ .

And the explicit formula would be  $a_n = 4n^2 + 13n + 1$ .

4. Think of a right triangle with lengths of sides as  $a = 5$ ,  $b = 4$ ,  $c = 3$ . Use these lengths to construct a right triangle and then construct squares on each of the sides as shown below. Next create right triangles similar to the original on the squares made from the legs. And repeat the process. In the resulting diagram seen below, what is the area of blue squares; what is the area of green squares?





### Analysis of the problem:

The blue squares are similar with sides in the ratio  $c/a$ . Thus the ratio of the areas of these similar squares must be  $(c/a)^2$ , or  $c^2/a^2$ . Hence the areas of blue squares appear to form the sequence below:

$$A_1 = 3^2 = c^2$$

$$A_2 = (9/5)^2 = (c^2/a)^2 = c^4/a^2$$

$$A_3 = (27/5)^2 = (c^4/a^2)^2 = c^6/a^4$$

...

Recursive formula:  $A_{n+1} = (A_n)(c^2/a^2)$

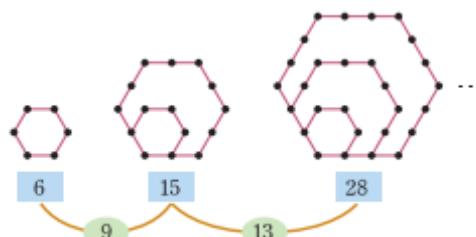
And in general, a closed form rule for  $A_n$  is  $c^{2n}/a^{(n+1)}$

Note that the technology used in creating this example is both Java and Geometer's Sketchpad using recursion. The Java (turtle) was created by Hanhyuk, Cho, Professor, Seoul National University.

5. Figurate Patterns: Use dots to create a regular hexagon with side of length 1 as shown in the first term. Successively increase the length of the sides of the hexagon to obtain the next figures. How many points are in the  $n$ th term?



### Analysis of the problem:



We use recursion as follows:

$$a_1 = 6$$

$$a_2 = a_1 + 2 \times 4 + 1 = 6 + 9 = 15$$

$$a_3 = a_2 + 3 \times 4 + 1 = 6 + 9 + 13 = 28$$

$$a_4 = a_3 + 4 \times 4 + 1 = 6 + 9 + 13 + 17 = 45$$

...

$$\begin{aligned} a_n &= a_{n-1} + 4n + 1 \\ &= a_{n-2} + 4(n-1) + 1 + 4n + 1 \\ &= \dots \\ &= a_1 + 9 + 13 + 17 + \dots + 4n + 1 \\ a_n &= (n+1)(2n+1) \end{aligned}$$

#### Proof by mathematical induction:

We know that  $a_1 = (1+1)(2+1) = 6$  which fits the pattern

$$a_n = (n+1)(2n+1) \text{ when } n = 1.$$

Suppose that the pattern is true for some  $k$  so that

$$\begin{aligned} a_k &= 6 + 9 + 13 + 17 + \dots + 4k + 1 \\ &= (k+1)(2k+1) \end{aligned}$$

Then,

$$\begin{aligned} a_{k+1} &= a_k + 4(k+1) + 1 \\ &= 6 + 9 + 13 + 17 + \dots + 4k + 1 + 4(k+1) + 1 \\ &= (k+1)(2k+1) + 4(k+1) + 1 \\ &= 2k^2 + 7k + 6 \\ &= (k+2)(2k+3) \\ &= [(k+1)+1][2(k+1)+1] \end{aligned}$$

Thus the generalization for  $k+1$  is true for any  $k$  for which it is true and hence the generalization is true for all values of  $k$ .

#### Spreadsheet Applications

6. When using a spreadsheet, recursive processes often occur. The spreadsheet illustrated here has three recursive functions in columns A, B, and C:
- A:  $f_n = f_{n-1} + 1$
  - B:  $f_n = f_{n-1} + 300$
  - C:  $f_n = 1.03f_{n-1} + 300$

Application:

A: years

B: total amount, without interest

C: total amount, with 3% interest per year.

	A	B	C	D	E
1	0	700	700		
2	1	1000	1021		
3	2	1300	1352		
4	3	1600	1692		
5	4	1900	2043		
6	5	2200	2404		
7	6	2500	2776		
8	7	2800	3160		
9	8	3100	3554		
10	9	3400	3961		
11	10	3700	4380		
12					
13					

7. The Fibonacci-sequence  $f_{n+2} = f_{n+1} + f_n$  can be unfolded in a spreadsheet where A1=1 and A2=1.

	A	B	C
1	1		
2	1		
3		2	
4		3	
5		5	
6		8	
7		13	
8		21	
9		34	
10	=A9+A8		

8. Consider  $1 + \frac{x}{1} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!}$ , the approximate value of  $e^x = \exp(x)$  where  $x$  is real and  $n$  is nonnegative integer. In order to calculate this expression, write it in recursive form as

$$u(0) = 1, v(0) = 1, v(n) = \left(\frac{x}{n}\right)v(n-1), u(n) = u(n-1) + v(n)$$

with precondition  $n \geq 0$ .

In order to calculate the value of  $u(n)$  for some integer value of  $n$ , we write an algorithm using the function in recursive form as follows:

```
function v(x: real, n: integer): real
begin
  if n = 0 then return 1
  else return (x/n) * v(x, n-1)
end

function u(x: real, n: integer): real
begin
  if n = 0 then return 1
  else return u(n - 1) + v(x, n)
end
```

Or we may write an algorithm using an iterative form as follows:

```
function f(x: real, n: integer): real
var u, v, i : integer
begin
  u = 1
  v = 1
  for i = 1 to n do
    begin
      v = (x/i) * v
      u = u + v
    end
  return u
end
```

Note that the algorithm using iteration uses a do-loop to repeat the computation until the desired result is obtained while the algorithm using recursion calls itself with smaller arguments until the base case is reached.

9. Calculate  $1 + \frac{x}{1} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!}$ , the approximate value of  $e^x = \exp(x)$ , where  $x$  is real and  $n$  is nonnegative integer. By replacing  $n$  with 0, 1, 2, 3, 4, ..., 11,  $x = 0.3$  and using a spreadsheet, we obtain the table below which shows the approximate value of  $e^{0.3} = \exp(0.3)$ .

	A	B	C	D	E	F	G
1	x=	0.3					
2		n	v(n)	u(n)			
3		0	1	1			
4		1	0.3	1.3			
5		2	0.045	1.345			
6		3	0.0045	1.3495			
7		4	0.0003375	1.3498375			
8		5	2.025E-05	1.34985775			
9		6	1.013E-06	1.34985876			
10		7	4.339E-08	1.34985881			
11		8	1.627E-09	1.34985881			
12		9	5.424E-11	1.34985881			
13		10	1.627E-12	1.34985881			
14		11	4.438E-14	=E13+D14			
15							
16							
17							