ASSESSMENT OF REASONING AND PROOF

Introduction

The nature of mathematical reasoning and proof is a defining characteristic that sets mathematics apart from other disciplines in terms of how knowledge and truth are viewed. Reasoning consists of all the connections between experiences and knowledge that a person uses to explain what they see, think and conclude. It is the process that underlies exploration and the discovery of new ideas. It also plays a central part in proof. By proof we understand not just the formal process of constructing logically consistent arguments based on axioms, definitions, and theorems traditionally found in school geometry courses but all the activity that leads to discovering a mathematical fact, establishing a conjecture and constructing a justification, including exploring, generalizing, reasoning, arguing, and validating. As such it is important that school students be exposed to mathematical reasoning and proof in this broad sense and that they gain an appreciation and understanding of the nature of reasoning and proof, and an understanding of the corresponding processes to a level appropriate to their mathematical development.

In this context, to ensure that sufficient importance is given to this dimension of mathematical education, all aspects of reasoning and proof must be assessed, both during (formative assessment) and at the end (summative assessment) of learning activities, with the following aims in mind:

1. Provide information about the level of competence achieved by students (summative).
2. Provide students with feedback about their learning, which can help them proceed with a problem (formative), identify their strengths and weaknesses, and set targets (summative).
3. Provide teachers with information that can inform classroom practice (formative).
4. Help curriculum developers and policy makers assess the quality of the mathematical education received by students in the domain of reasoning and proof (summative).

Summative Assessment

Testing under controlled conditions plays a central role in the assessment of students' mathematical achievement for two principal reasons: it provides a neutral measure of performance and is cost efficient. Given the widespread use of this mode of assessment it is important to make sure that reasoning is assessed. It is also necessary to assess as many of the elements as possible that are involved in understanding proof and developing the ability to prove. Included in these are recognizing the role of assumptions, comprehending dependency relations, deducing information from given information, mathematically visualizing a figure, including decomposing and recomposing figures, and reasoning from and about visual representations of mathematical objects with knowledge of the theoretical elements related to the situation.

- Summative assessment methods can evaluate students' knowledge and understanding under controlled conditions but still retain relevance and significance for students. An example of this is introduced in the French Baccalaureat exam in which the instructor poses a problem to four students and assesses their ability to explore and prove the problem. In this method of assessment if students ask for help, the instructor has a handout of hints from which they can choose.

Aspects of reasoning and proof that can be assessed effectively (but not exclusively) under controlled conditions include:

Use of mathematical reasoning

Students demonstrate their ability to reason mathematically by showing the steps taken in arriving at a solution. They should get credit for their work, which may be difficult on multiple choice tests.

Understanding proof

Students demonstrate that they understand the essential nature of mathematical proof through their answers to questions which require them to:

- complete the steps in a given proof (either establish the statement corresponding to a reason or provide the reason for a given statement)
- establish relationships between the steps in a given proof (identifying which of the previous steps in a proof are necessary to deduce the statement established in a step)
- find errors in a given proof
- evaluate the validity of a given proof
- compare and evaluate different justifications for a given problem (empirical explanations, proofs based on a generic example, proofs based on an axiomatic system)

Learning to prove

The construction of a proof under test conditions is a valid exercise but one which requires careful preparation. If it is the only way in which proof is assessed, it may result in students having a distorted and negative view of the processes by which mathematicians arrive at conclusions. An important factor to take into account is the previous knowledge of the students taking the test: if they have already seen the proof in question, then the assessment objective is invalidated. Alternative tasks that can be used to assess students' ability to construct proofs include asking them to:

- outline a proof
- identify the mathematical knowledge required for a given proof
- fill in missing steps in a given proof
- provide a set of hints for someone else to construct a proof
- adapt a given proof to a new situation in which one or more elements have been changed or the assumptions have been changed
- provide an alternative proof for a given situation
- provide a "local" proof (working within a self-contained subset of an axiomatic system)

To diminish the pressure students may feel when an assessment includes only the construction of proofs is to allow them to choose the problem on which they want to work from a given set of problems.

A mode of assessment that bridges the gap between summative and formative assessment and which is of vital importance in assessing reasoning and proof is project work. For students, this type of activity can exemplify the way mathematics is constructed; an open-ended problem is proposed that leads to exploration that results in the formulation of a conjecture and to its justification. Students can demonstrate their competencies relative to a set of criteria established.

Formative assessment

Assessment *for* learning, rather than *of* learning, focuses principally on process. Formative assessment, in which teachers seek to understand student thinking and use it to shape their instruction, helps students reflect on their own learning and the competencies they are developing.

One of the challenges of assessing the processes by which students develop their reasoning and argumentation is the non-linear nature of problem solving and investigation work. It is important to include variety and flexibility in the range of assessment techniques used and to recognise the inherently subjective nature of this type of assessment as a positive quality rather than a deficiency. For formative assessment to work, teachers must be confident in the use of formative assessment techniques and their professional judgement should be acknowledged and promoted in professional development.

Activities that can be used for the formative assessment of reasoning and proof include the participation of students in class discussions, presentations by students of their work and individual or group projects and investigations. Three phases of development can be identified (corresponding to the competencies found in the Western Australian *Curriculum Framework* (2007) progress maps), each of which can be assessed, although it is important to stress that in practice these are interconnected and do not follow a linear progression.

The creative phase

This is the phase in which students explore a problem or situation and develop their understanding of the problem. We can assess:

- their capacity to modify or restrict the problem to one they are able to tackle
- their ability to identify and express the assumptions they make in tackling the problem
- their use of technology or other means to explore the problem
- their capacity to make conjectures based on the exploration of the problem

The reasoning and argumentation phase

In this phase students start to test their conjectures and look for reasons to support or refute them. They might generate further examples using technology or other means in order to guide their reasoning towards a justification or validation, or towards a reformulation of the conjecture. We can assess:

- their powers of analysis and synthesis
- their deductive and inductive reasoning
- the generation of examples to test their conjecture
- the identification of "blind alleys" and their responses to these in reformulating conjectures

The justification and validation phase

In this phase students construct proofs of their results, and the validity of these proofs can be assessed.

Examples of different types of assessment items are given in the appendix.

References

For both summative and formative assessment, guidelines can be found at

- http://www.curriculum.wa.edu.au/ProgressMaps/Documents/Mathematics/Working%20Mathematically_1.doc This document is the *Progress Maps for the Working Mathematically Strand of the Mathematics Curriculum* in Western Australia.

Another possible guideline for assessment can be found at:

- Description de l'expérimentation 2006-2007 sur Éduscol et banque de descriptifs de sujets http://eduscol.education.fr/D1115/epr_pratique_presentation.htm
- Les sujets complets proposés aux élèves http://www.apmep.asso.fr/IMG/pdf/Feleves.pdf
- Rapport de l'inspection générale sur l'épreuve pratique http://www.education.gouv.fr/cid4909/experimentation-d-une-epreuve-pratique-de-mathematiques-au-baccalaureat-scientifique.html

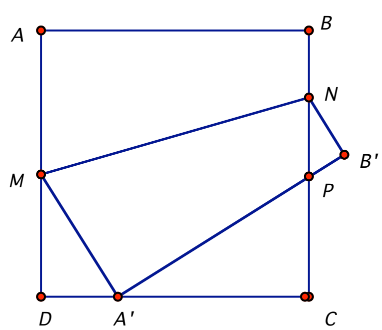
Appendix

Examples that can be used to construct assessment tasks such as those described in the brief are given below. While students may use inductive reasoning to make conjectures, they must use deductive reasoning to justify their own or other students' conjectures. In terms of assessment it is important for students to also report hypotheses that turned out to be not true and explain why.

Example 1

Begin with a square piece of paper labelled *ABCD* as in Figure 1. Fold the paper so that the point *A* lies on segment *CD*. Move the point *A* back and forth on segment *CD*. Make conjectures from what you see happening. This problem was posed to teachers at the Park City Mathematics Institute Secondary School Teachers Program. More information can be found at http://mathforum.org/pcmi/hstp/sum2007/morning/.

Figure 1: Folded square of paper



When square origami paper *ABCD* is folded as shown in Figure 1 with point *A* folded to point *A′*, three triangles, *A′DM*, *A′CP* and *B′NP* are formed. With this drawing, many conjectures at varying levels of difficulty are possible, for instance:

- segment *MA′* is perpendicular to segment *B′N*
- triangle *A′DM*, triangle *PCA′* and triangle *PB′N* are similar

The many possible conjectures allow the instructor to use a problem that students can configure to their own level of mathematical development. While exploring this problem, students can become aware of the mathematical properties required to prove their conjectures. This problem also lends itself to local proof and/or adapting the proof; for example, the students might examine the situation for a rectangle instead of a square.

Students can be tested with respect to the comprehension of the properties both given and discovered if they are required to construct a representation of the figure using a dynamic geometry program.

Example 2

> For a given natural number *n*, determine an additive decomposition of *n* into two natural numbers, *a* and *b*, so that their product, *ab*, is the greatest value possible.

The level of knowledge students have determines the process they would use to formulate a conjecture: when the difference between the two numbers is the least, the product is the greatest.

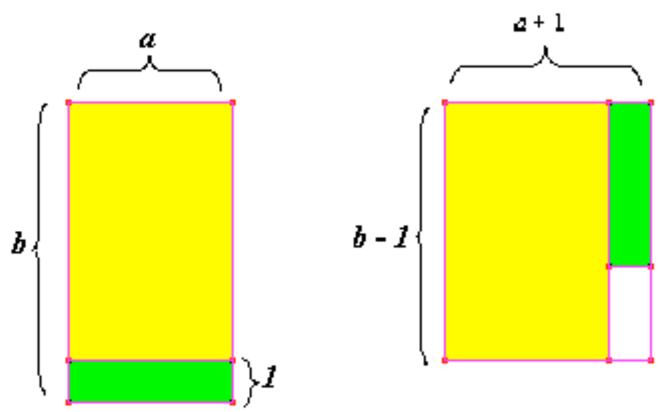
- Students can give examples. For 10, the decomposition is 5 and 5. For 13, the decomposition is 6 and 7.
- Students can use a graphing calculator to represent the relationship between the difference and the product, either using a table or graphing the ordered pairs (i.e., if $a - b = 10$, then $a = 10 - b$ and $ab = c$; thus, you are looking for the maximum in $a(10 - a) = c$.

The proof can be tested with a visual proof "form", as in Figure 2.

- Explain how Figure 2 can be used to justify the following theorem.

  Theorem If a and b are natural numbers with $a < b$, then $ab \leq (a + 1)(b - 1)$.

  Figure 2: Visual Demonstration of Multiplication of Binomials



- How can you use this theorem to justify the conjecture?

The problem can be generalized and a guided process carried out which involves reasoning and proof, using the following statement;

For a given natural number n, consider an additive decomposition, a and *b, with natural numbers so that the product has the greatest value possible.*

- Explain why in any additive decomposition of a number which contains a 5, a greater product is obtained when the 5 is replaced by 3 + 2.
- Explain why in any additive decomposition of a number which contains a number greater than 5, a greater product is obtained when it is replaced by a sum that contains only 2, 3 or 4.
- Explain why when there are more than two 4's in the decomposition, it is better to replace each pair of 4's by an additive decomposition with numbers less than 4.