Institute for Advanced Study/Park City Mathematics Institute International Seminar: Bridging Policy and Practice in the Context of Reasoning and Proof
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Preparation of Teachers for Teaching Problem Solving, Reasoning and Proof

Mathematicians and mathematics educators generally agree that the mathematical processes of problem solving, reasoning, and proof are integral to the teaching and learning of mathematics. As such, the preparation required of teachers to teach these processes is not trivial. Most countries have some type of teacher preparation program; however, most of these programs do not explicitly prepare teachers to teach problem solving, reasoning, or proof. Furthermore, practicing teachers may not possess the skills, knowledge, or attitudes to effectively teach these processes.

In an attempt to bridge practice and policy, participants of the Park City Mathematics Institute's International Seminar offer characteristics as well as knowledge teachers must have so they are prepared to plan and teach students how to problem solve, reason, and prove in a mathematics classroom.

A. Necessary Components of Teacher Education

1) Knowledge

To help illustrate each knowledge type identified below, an example has been included at the end of this section.

i) Mathematical Content Knowledge

Recognizing that grade levels vary around the world, the divisions are made for discussion purposes only. The content courses reflect what may be the minimum requirement in some countries and the maximum content in others. The content knowledge presented is meant to frame general guidelines to help identify what mathematical content knowledge is required for teachers to be best prepared to teach problem solving, reasoning, and proof in the mathematics classroom.

Primary (grades 1-5) Content Knowledge
Teachers of primary students should possess a profound understanding of school mathematics up to but not necessarily including calculus from a higher point of view, with an emphasis on foundations of mathematics, arithmetic, geometry, elementary probability, and statistics.

Lower Secondary (grades 6-8) Content Knowledge
Teachers of lower secondary students should possess all the primary content knowledge indicated above, plus algebra, calculus in one dimension, advanced geometry, and trigonometry.

Upper Secondary (grades 9-12) Content Knowledge
Teachers of upper secondary students should possess all the lower secondary content knowledge indicated above, plus calculus in several dimensions, differential equations, further knowledge of probability and statistics, linear algebra, and matrices.

ii) Pedagogical Knowledge

Teachers at all levels should have knowledge of child development, general knowledge about teaching methods, educational theories, knowledge of diversity (e.g. language, gender, ethnicity), and theories of learning. Knowledge of child development theories is necessary across all educational levels.

iii) Didactical Knowledge (Mathematical Pedagogical Knowledge)

To teach problem solving, reasoning, and proof, teachers must possess subject related didactical knowledge (e.g., knowledge about teaching of geometry). An understanding of psychological theories about teaching mathematics (e.g., constructivism, social cultural and social critical theories as they relate to the teaching of mathematics) and knowledge about the processes involved in how students think about problem solving, reasoning, and proof are also required. Teachers must also have the knowledge of teaching methods to be able to facilitate the teaching of problem solving, for example, be able to apply Polya's principles for problem solving (1957). To effectively teach proof, it is critical that teachers know what constitutes a valid proof at appropriate grade levels for their students. Furthermore, teachers must also be aware and knowledgeable about the various levels of proof, ranging from pre-formal to formal, and when and how each level can be used adequately. Lastly, teachers must recognize and honor the fact that students interact in different ways with different mathematical representations (visual, symbolic, numeric).

Technology plays a crucial role in the learning of mathematics, especially as it relates to problem solving. Teachers should have proficiency with various kinds of mathematical software (e.g., dynamical geometry software, computer algebra systems, spreadsheets), and they should be able to use the technology in a variety of mathematical contexts. For example, technology could be used as a tool for visualization, modeling, or as means to support discovery learning.

Following is an example to help clarify the knowledge types identified above.

Example:

"Explain why the sum of two even numbers is even. "

Mathematical Knowledge: One component of the mathematical knowledge required for this problem is an understanding of foundations of number properties and the knowledge of what constitutes proof for the given example. This knowledge would also include recognition that different definitions of even are possible (pairing without a remainder, have a factor of 2, 2n for any whole number n).

Pedagogical Knowledge: Teachers would need to know effective classroom management, learning methods, and ways of thinking. When a teacher presents students with the problem, a pedagogically knowledgeable teacher will not merely demonstrate a solution but might instead design an activity using cubes and guided questions to help students "discover" why the sum of two even numbers is even.

Didactical Knowledge: Teaching might involve the decision to use concrete manipulatives to help students understand the problem or to use the number line as a tool. Teachers should recognize and connect multiple solution methods and representations. Teachers also need to know what constitutes a valid proof at a level appropriate for their students in order to be able to react adequately to different proposals developed by the students or in order to facilitate different proposals by the students. A teacher's didactical knowledge would also be shown in her ability to connect an explanation of pairing cubes to the demonstration that the sum of two even numbers is even with a more abstract explanation of $2n+2n=4n$.

2) Beliefs, Attitudes, and Values

Teachers hold beliefs about both the subject of mathematics and the teaching of mathematics. Some teachers believe mathematics is about discovering and experimenting. Others believe that mathematics is about proving and deducing mathematical truths. Both perspectives together comprise the whole of mathematics, and teachers need to be aware of both faces and not reduce mathematics to just one or the other.

When it comes to beliefs about the teaching of mathematics, teachers must have an open mind about both the teaching and learning of mathematics. That is, learning and teaching of mathematics does not happen at only an abstract level. Learning is a multi-dimensional process. It is important for teachers to hold the belief that teachers and students learn from each other. Furthermore, teachers must recognize the inherent value of problem solving, reasoning, and proof as part of the learning process. Finally, to teach these processes, a teacher must recognize that modeling and discovery learning are important

approaches in mathematics teaching and that mathematics should be taught with an emphasis on using activity-oriented approaches.

Teachers should be able to reflect critically on both the mathematics they teach as well as the teaching of that mathematics.

3) Volition

The effective instruction of problem solving, reasoning, and proof requires that teachers possess a willingness and desire to put into practice the above beliefs and knowledge. In order to do that, teachers must be able to connect the different kinds of knowledge (mathematical knowledge, pedagogical knowledge, and didactical knowledge) by using higher order skills, such as critical thinking, creative thinking, and self-regulated thinking (sometimes referred to as meta-cognitive competencies or proto-competencies). They must also be willing to change their beliefs and practice to reflect best practice as revealed by current research. That is, they must be dedicated to the concept of themselves as life long learners. For the sake of student learning, it is imperative that teachers encourage and acknowledge different student solutions as well as different approaches to proving mathematical claims.

B. How do we prepare teachers to teach problem solving, and reasoning and proof?

The preparation of teachers of mathematics is always of significant concern. However, to teach problem solving, reasoning, and proof, particular attention must be paid to teacher training. Teachers of mathematics are typically given opportunities to learn the mathematics but not necessarily in a problem-solving environment. Therefore, it is important that teachers have the opportunity to experience problem solving, reasoning, and proof first hand from the perspective of a learner.

i) Pre-service Teachers

Future teachers need the opportunity to experience problem solving, reasoning, and proof with close connection to school practice. Ample opportunities for work in actual classrooms are required to help pre-service teachers gain experience in transferring mathematical knowledge about problem solving into the classroom setting. Additionally, mathematical knowledge, pedagogical knowledge, and didactical knowledge should be a part of teacher education.

ii) In-service Teachers

Reflection is critical to the professional growth of a teacher. Therefore, practicing teachers should be provided the opportunity to reflect on their own teaching, preferably with others, concerning problem solving, reasoning, and proof.

Specifically, teachers need training and experience in problem development, reasoning, and proof according to their educational level and mathematical content knowledge. One effective way to help practicing teachers develop these skills is to create teacher collectives. These are groups of teachers, either in the same school or in nearby areas, that regularly come together to solve problems, share ideas, discuss pedagogy, plan lessons, and reflect collaboratively about teaching and learning. Other ways to help develop knowledge and skills in problem solving, reasoning, and proof could be the use of peer observation and team teaching. Mentoring and other support measures should be established on a broad basis. Lastly, school-based working groups can be used to provide in-service training for teachers related to these areas in the mathematics curriculum.

We have attempted to provide a framework for preparing teachers to teach problem solving, reasoning, and proof. We recognize that there are many constraints and issues that differ from country to country, but the beliefs, knowledge, and training we have identified are critical to properly prepare teachers to be confident and successful in the classroom.

References

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