The Place of Functions in the School Mathematics Curriculum

Functions are used in every branch of mathematics, as algebraic operations on numbers, transformations on points in the plane or in space, intersection and union of pairs of sets, and so forth. Function is a unifying concept in all mathematics. Relationships among phenomena in everyday life, such as the relationship between the speed of a car and the distance travelled, are functions. The concept of function has an important part in the school mathematics curriculum; yet, many countries today are concerned with how to structure the curriculum. The first part of this brief provides a short background about the historical development of function in mathematics and its introduction into the school curriculum, explicating the 'identity crisis' many countries are facing today. We then elaborate more on the goals for including functions in the school curriculum and the learning about functions. We conclude this brief with challenges that policy makers, curriculum developers and teachers face and possibilities for meeting these challenges.

Introduction

The concept of function has undergone an interesting evolution. Developments in mathematics have changed the concept of function from a curve described by a motion (17th century) to an analytic expression made up of variables and constants representing the relation between two or more variables with its graph having no "sharp corners" (18th century). In the 19th and 20th centuries, new discoveries and fresh emphasis on rigour led to the modern conception of a function as a univalent correspondence between two sets. More formally, a function f from A to B is defined as any subset of the Cartesian product of A and B, such that for every $a \in$ A there is exactly one $b \in$ B such that $(a, b) \in f$.

As the discipline of mathematics has grown, function has become one of the most important and fundamental mathematical concepts as a way to organize and characterize mathematical relationships. The development of the concept of function in mathematics influenced the way function entered and was presented in school mathematics. The function concept appeared in secondary mathematics curriculum at the beginning of the 20th century, amplified by the creation of the International Commission on Mathematical Instruction (ICMI) in 1908. The term 'function' was used then in a way similar to Euler's definition from the 18th century. Function, in the modern sense, was introduced to the school curriculum in many countries during the 'new math' reform of the 1950s-1960s, often using the function concept as an organizing theme for the secondary mathematics curriculum. Today, the modern concept of function appears in many school mathematics curricula. A common expectation is that at the end of high school, students will know the function concept in general and be familiar with specific types of functions, including linear, quadratic, general polynomial, reciprocal, power, step, exponential, logarithmic, trigonometric, and piece-wise functions in different representations. Yet, numerous studies reveal that students have difficulties learning the concept of function. And, although function is often defined in textbooks in a modern sense, students tend to hold a restricted image of function, similar to the one from the 18th century when function was a dependent variable or an analytic expression whose graph has no 'sharp points'.

In addition to the problems arising from students' restricted images of the range of the notion of function, there is evidence that the different representations of functions often stand and are treated in isolation, without the connections that make them representations of a common core concept. Moreover, there seems to be a growing uncertainty amongst curriculum designers, textbook authors, teachers and others about the purposes and goals of the study of functions, in particular as regards the relative emphasis on intra-mathematical goals and goals related to applications and modelling. All this gives rise to what we would like to call an 'identity crisis' in the teaching and learning of functions in school.

The term ‘curriculum’ has multiple meanings. In this brief, we use it in a broader sense to include the rationale, goals and intentions, principles and standards, the syllabus, course outlines, textbooks, teacher guides, and other learning, teaching and assessment materials, the mathematical content studied, the teaching and assessment methods. So, for us, curriculum includes what is taught as well as how it is taught; how what is taught is assessed and for what purposes. We also note that the official curriculum is not identical to how intentions are translated and expressed in curriculum materials (e.g., textbooks and teacher guides), in tests and examinations, or in what actually happens in classrooms.

Some countries or states have an official curriculum whereas others do not. Yet, even in countries that do have an official national mathematics curriculum, the goals of the curriculum may be different for different groups of students.

Goals for functions

There are several different purposes for including functions in the curriculum. While the first three purposes are internal to mathematics, we note that the fourth is a different kind of purpose.

- Include function as a mathematical topic that is perceived as an intrinsic part of mathematics in its own right. For some students this may be because functions are likely to occur in their later studies.
- Introduce the concept of function as a unifying concept across the entire mathematical curriculum. For instance geometric transformations of the plane can be perceived and investigated as functions. The same is true of arithmetic operations, some solution sets to equations, formulae used in mensuration (length, area, and volume), probabilities, regression functions, recursive definition of objects, etc.
- Use functions as a vehicle for clarifying mathematical thinking and reasoning, as a tool for proving statements, and suchlike.
- Use functions to provide a means for extra-mathematical ends, in particular to represent, describe and deal with (i.e., to model) phenomena, situations, or problems outside of mathematics itself. In some countries and states, this purpose is a key driver of the curriculum, provoking the identity crisis referred to above.

We believe it is essential for curriculum designers, textbook authors, teachers, teacher educators, and others, to clarify which of these purposes are to be pursued in a given programme or teaching context, as the purposes are going to shape and influence in a crucial manner what is taking place in teaching and learning. In particular it is important to consider whether the needs of different groups of students should give rise to the pursuit of different purposes for these groups.

When looking at the current state and development of upper secondary curricula around the world, it seems that a dichotomy is emerging between the adoption and pursuit of a modelling purpose, on the one hand, and an intrinsically mathematical purpose, on the other hand, for the study of functions. We believe that such a dichotomy is unfortunate and can be avoided. Therefore, efforts should be made to strike a proper balance between the two kinds of purposes.

Learning about functions

Once goals are chosen, decisions need to be made about the structure of teaching and learning and phasing activities over the years of secondary school in order to achieve them. School curricula around the world have taken different approaches to including functions in the curriculum. Key decisions that need to be made concern the way in which functions are defined formally, when this occurs in a developmental sequence and the variety of ways in which students encounter functions, consistent with the goals chosen.

It is clear that the term ‘function’ has a range of meanings in everyday speech, differing from culture to culture, often not compatible with and at any rate much less precise than a mathematical definition. Examples of these include function as purpose: “The function of the brake is to stop the car”, function as event: “The function to celebrate the school’s sporting victory will be held on Wednesday”, function as role: “His function on the committee was to take notes”, and function as mechanism: “The function of the switch is to turn on the light.” Such everyday meanings of the term may impede students’ development of the specific mathematical meanings. In some languages, the term ‘function’ may not appear at all.

The relationship between the definition of function and the mental image students develop of function is important and has been studied by researchers (e.g., Vinner 1983). There are at least three possible approaches to the inclusion of functions in the school curriculum, all of which have been used in various countries over the past few decades:

- Students are provided with experiences with classes of functions (such as linear, quadratic (polynomial), reciprocal, exponential, logarithmic). This experience is then drawn upon to construct a general definition, later in secondary school.

- Students are given a general definition of the concept of a function, so that later experiences of classes of functions can be interpreted as special cases of the function concept.

- Students are given experiences with various classes of functions as important objects in their own right, but a general definition of the concept is not provided at all in secondary school.

While each of these approaches has advantages and enthusiastic supporters, there are also some pitfalls for each that have been recognised in practice and in research. For example, the first approach above may lead to students acquiring a very limited concept image for function, such as one that is restricted to continuous functions that are easily expressed in algebraic symbols or for which there is a two-dimensional graph. A pitfall of the second approach is that students may have at first such little experience of the concept of a function that the definition does not have much meaning for them and is not used to interpret their later work. The third approach runs a risk that students have too little opportunity to see the general concept of a function, unless it has been drawn to their attention in some way, although some would argue that this is not problematic for some students (such as those who do not continue studying mathematics beyond secondary school).

Irrespective of the approach that is chosen, it is important to note that the seeds of the secondary school work on functions are laid in the first several years of school. An example of this is the study of patterns in elementary schools, where students are helped to observe and describe relationships between quantities.

There are various notations for functions, and it seems important for students to eventually be comfortable with most of these, rather than being restricted to only one. Indeed, students who understand functions with only one form of notation are unlikely to have an adequately inclusive concept image. Nevertheless, in some places, a preference might be expressed for a single notation in order to avoid any confusion for students. Common notations include:

- $f(x) = x^2 + 1$

- $y = x^2 + 1$

- $y = f(x) = x^2 + 1$

- $f: x \rightarrow x^2 + 1$

- $(x,y) \in \{y = x^2 + 1\}$

The first of these draws attention to the functional nature of the relationship, while the second supports a graphical interpretation in the plane. The third notation emphasises the functional character of the graph, while the fourth is consistent with the idea of a mapping from one set to another, and the fifth emphasises the idea of a set of points. Over the course of secondary school, students might progressively encounter this variety of notations.

As well as notations, students' concept images are influenced by the representations of functions that they experience. To develop a rich concept image—which seems important to develop a rich meaning for and use of the concept—students ought to encounter functions in different representations and make connections between these (Thompson, 1994). Some representations support particular ways of thinking about functions especially well. Among the representations that seem important to include and which students might progressively encounter are:

- Symbolic (e.g. an algebraic rule such as $G(a) = 3a - 1$)
- Graphic (e.g. a graph of the quadratic function $y = 3 - x^2$)
- Diagrammatic (e.g., a mapping diagram)
- Verbal (e.g. a description such as "$40 initial charge and $60 per hour" for the labour fee for a plumber.)
- Tabular (e.g. a table showing the population of Chile each year)
- Implicit (e.g., functions emerging from parameterising solution sets of equations)

The study of functions at school allows for the later development of functions as tools for more advanced study of mathematics. Some of the more sophisticated uses of functions to support advanced mathematical thinking are included in school curricula in various countries, but there is not space here to elaborate on the details. Some of these more sophisticated ideas related to functions include: operations with functions, composition of functions, inversion, differentiation, integration and differential equations and properties of functions (such as montonicity, injectivity, boundedness, covariance and optimisation). Typically, these are addressed in the final years of secondary school, if at all, or left to the early undergraduate years.

Teaching of functions

While there is a potential identity crisis for functions in the curriculum, teachers in classrooms still have to make decisions on what to do. Such decisions are always complex and challenging and are made more difficult when the goals for including functions in the curriculum are not clear. Thus there are many challenges to be addressed by teachers.

One challenge is that textbook writers do not present the notion of function in ways that connect function to the rest of the curriculum, leaving this task to the teacher. The teacher has to ensure that eventually students develop a concept image that matches a comprehensive definition of function. Another challenge is that students tend not to make connections among different representations and are unable to extract the underlying core concept. Yet another challenge for teachers is to take the time to allow students to explore and develop deep and rich understanding of function. While the use of technology can help, even then the optimal allocation of classroom time provides a difficult challenge.

No royal road has been identified internationally as the best way for teaching and presenting function to students, but all successful methods address the various challenges in a conscious well-thought and systematic manner. Good teachers provide rich environments, emphasizing making connections among different representations, and allow students to take the time to develop such understanding. Some countries or places do not adopt such a thoughtful approach, yet, there are quite a few countries and places that do that.

A perennial challenge for teachers everywhere involves motivating students, and the teaching of functions is no different in that respect. Not all students are alike, so some may be motivated by seeing that functions are everywhere, in and outside of mathematics; others may be motivated by working on challenging tasks or in seeing surprising connections. Indeed, it seems likely that one of the reasons for the widespread popularity of a ‘modelling approach’ to teaching functions is that such an approach helps students appreciate that mathematics has some practical application in the everyday world.

In recent years, the role of technology has been explored in a number of ways, depending on the facilities available to teachers. It seems clear that technology provides good opportunities to support student learning about functions, and it seems likely that this will continue to be an area of some promise for teachers. Many of these opportunities involve students in exploring functional ideas for themselves. Some potential benefits and opportunities include:

- The use of multiple representations of functions on computers and calculators
- Facilitation of the use of functions for modelling purposes
- Software platforms for explorations of properties of functions
- Software platforms for consolidating function as object (e.g., families of functions)
- Manipulative tools, including computer algebra systems, to avoid tedious and extensive symbolic manipulation
- Environments, such as dynamic geometry environments, for exploring other kinds of functions: such as those mapping $R^2 \rightarrow R^2$ (Hazzan & Goldenberg, 1994)

There are also some pitfalls associated with the use of technology identified by researchers and practitioners. These include at least

- Concept images can be restricted by technology, which can often best represent functions with distinctly numerical features.
- A balance of human activity and understanding and use of computers is needed to avoid technology being used as a ‘black box’.
- It takes time for students to learn the software tools, and time is always in short supply in classrooms.

These and other aspects of the use of technology in the teaching and learning of functions are treated in more detail in a companion brief (PCMI International Seminar, 2009).

Especially in countries with high-stakes examinations, many teachers and researchers have reported on the powerful backlash effects of examinations on teacher decisions in the classroom. For example, when students are not permitted to use technology tools in exams, it seems unlikely that they will be widely used in teaching, and thus schools will have limited motivation to find the necessary resources. Even when technology is permitted or provided, it seems that it is not always used by teachers and schools; however, this effect is not by itself sufficient to explain the limited role technology plays in many curricula. By their nature, examinations frequently focus on a small range of predictable activities, often of a semi-routine nature, which may also have the effect of narrowing student experience and thus their concept images. Similarly, the reluctance to permit technology use in examinations may discourage more adventurous and helpful use of technology in classrooms.

Conclusion

Although functions appear in all school curricula in the secondary years, clarity of purpose for this continues to be needed, so that explicit advice can help teachers in their choices about what to teach and why. Decisions need to be made regarding the relative importance of the formal definition of a function and the practical applications of functions for modelling, as well as how to phase this work over the secondary years. The present situation in many countries has resulted in a kind of identity crisis for the place of functions, and it would be helpful if this were to be resolved. While technology holds some promise to support teaching and learning, it is no panacea, and care is needed to ensure that it is used productively, consistent with available facilities in different contexts, and further that the intentions for including functions in the curriculum are not unwittingly undermined by assessment practices.

References

Akkoç, H., & Tall, D. (2005). A mismatch between curriculum design and student learning: the case of the function concept. In D. Hewitt & A. Noyes (Eds), *Proceedings of the Sixth British Congress of Mathematics Education* held at the University of Warwick, pp. 1-8. Available from www.bsrlm.org.uk.

Hazzan, O., & Goldenberg, E.P. (1997). Students' understanding of the notion of function in dynamic geometry environments, *International Journal of Computers for Mathematical Learning*, 1, 263-291.

PCMI International Seminar Brief. (2009). *Assets and Pitfalls in Using Technology in Teaching and Learning Functions.*

Thompson, P. W. (1994), Students, functions, and the undergraduate curriculum. In *Issues in Mathematics Education*, (4), 21 - 44. Washington DC: College Board on Mathematical Sciences.

Vinner, S. (1983) Concept definition, concept image and the notion of function, *International Journal for Mathematical Education in Science and Technology*, 14(*3*), 293-305.