Problem Set 1: Shuf'ling

Welcome to PCMI. We know you'll learn a lot of mathematics here—maybe some new tricks, maybe some new perspectives on things with which you're already familiar. A few things you should know about how the class is organized.

- **Don't worry about answering all the questions.** If you're doing that, we haven't written the problem sets correctly.
- **Don't worry about getting to a certain problem number.** Some participants have been known to spend the entire session working on one problem (and perhaps a few of its extensions or consequences).
- **Stop and smell the roses.** Getting the correct answer to a question is not a be-all and end-all in this course. How does the question relate to others you've encountered? How do others think about this question?
- **Respect everyone's views.** Remember that you have something to learn from everyone else. Remember that everyone works at a different pace.
- **Teach only if you have to.** You may feel the temptation to teach others in your group. Fight it! We don't mean you should ignore people but give everyone the chance to discover. If you think it's a good time to teach quadratic reciprocity, think again: the problems should lead to the appropriate mathematics rather than requiring it. The same goes for technology: problems should lead to appropriate uses of technology rather than requiring it.
- **Each day has its Stuff.** There are problem categories: Important Stuff, Neat Stuff, Tough Stuff. Check out Important Stuff first. All the key mathematics in the course can be found and developed in Important Stuff. It's Important Stuff! Everything else is just Neat or Tough. If you didn't get through the Important Stuff, we noticed . . . and we'll change the course (yes, literally) to account for it. Every problem set is based on what happened in the previous set, and what happened in the previous *class*.

When you get to Problem Set 3, come back and read this introduction again.

Some of the problems have yet to be solved. Those are the *really* fun ones.

Do something really interesting, and maybe everyone else will be asked to try it the next day! That also means we want to hear about what you're doing.

Will you remember? Maybe . . .

Opener

Let's watch a video. Don't worry, it's only like 2 minutes long.

Wait *what?* Figure out what you can about this.

What is this I don't even.

Important Stuff

1. Does the perfect shuffle work for other deck sizes? If not, why not? If so, what stays the same and what changes?

2. Evelyn is thinking of a positive integer, and because she's a math teacher she calls it x. What information would you know about x based on each statement?

- **a.** $3x$ has last digit 4
- **b.** $7x$ has last digit 4
- **c.** $4x$ has last digit 4
- **d.** $5x$ has last digit 4

3. **a.** What number is $9 \cdot 10^1 + 9 \cdot 10^0 + 4 \cdot 10^{-1} + 4 \cdot 10^{-2}$?

b. Ben's favorite number is 802.11_{10}. Write it as a sum of powers of 10.

The little 10 here means the number is in base 10. Bases will generally be given when they seem needed, and we'll try not to be confusing.

4. **a.** What number is $1 \cdot 3^3 + 0 \cdot 3^2 + 2 \cdot 3^1 + 0 \cdot 3^0 + 1 \cdot 3^{-1}$?

b. Carol's favorite base-3 number 2110.2_3. Write it as a sum of powers of 3.

c. Convert 2110.2_3 to base 10.

Surely you came to PCMI armed with your favorite number in each base.

5. Write each number as a decimal. Write each number as a decimal. Write each number as a decimal.

These directions either terminate or repeat.

- **a.** $\frac{1}{2}$
- **b.** $\frac{1}{50}$
- **c.** $\frac{1}{9}$
- **d.** $\frac{2}{9}$
- **e.** $\frac{9}{9}$
- **f.** $\frac{1}{13}$

6. Hey, we just met you, and this is crazy; but here's some numbers, so make them base three.

We missed you so bad. We missed you so, so bad.

a. 9
b. 13
c. 242
d. $\frac{1}{9}$
e. $\frac{1}{13}$
f. $\frac{1}{242}$

7. Write each number as a base-3 "decimal".

a. $\frac{1}{9}$
b. $\frac{1}{2}$
c. $\frac{2}{2}$
d. $\frac{3}{2}$
e. $\frac{1}{13}$
f. $\frac{1}{242}$

Neat Stuff

8. Under what circumstances will a base-10 decimal repeat?

9. Under what circumstances will a base-3 decimal repeat?

10. The repeating decimal $.\overline{002}$ means .002002002 . . . . But what number is it? That depends on the *base*! Ace this problem by finding the base-10 fraction equal to $.\overline{002}$ in each given base.

Psst: You did some work on base 3 already. Ace this base problem, and you'll see the sine.

a. base 3
b. base 4
c. base 5
d. base 7
e. base n
f. base 2?!

11. We overheard Sara and Joe debating about whether or not the number .99999 . . . was equal to 1. What do you think? Come up with a convincing argument, and if you already know one, come up with a different one!

We didn't hear who was arguing each side, we mostly just ran away.

12. **a.** Find all positive integers n so that the base-10 decimal expansion of $\frac{1}{n}$ repeats in 3 digits or less.
b. Find all positive integers n so that the base-3 "decimal" expansion $\frac{1}{n}$ repeats in 4 digits or less.

13. Write 13 and 242 in base $\sqrt{3}$ instead of base 3. Hee hee hee. Or maybe this turns out to be totally awesome!

Tough Stuff

14. Aziz has a cube, and he wants to color its faces with two different colors. How many different colorings are possible? By “different” we mean that you can’t make one look like the other through a re-orientation.

15. What about edges?

16. **a.** Convert 13 to base $\frac{3}{2}$.
b. Convert 13 to base π.

Problem Set 2: Slhi'unfg

Opener

Can perfect shuffles restore a deck with 9 cards to its original state? If so, how many perfect shuffles does it take? If not, why not?

Today's title comes from the German word for "shuffling".

Split the cards 5-and-4, and keep the top card on top.

Important Stuff

1. Working with your table, fill in a whole lot of *this* table:
`http://www.tinyurl.com/perfectshuffle`

The file is *in* the computer! Oh, only one computer per table, please.

2. Find the units digit of each annoying calculation. Put those calculators away!

a. $2314 \cdot 426 + 573 \cdot 234$
b. $(46+1)(46+2)(46+3)(46+4)(46+5)$
c. $71^4 \cdot 73^4 \cdot 77^4 \cdot 79^4$

The *units digit* of 90210 is 0, matching Brenda's IQ.

You down with ZPP? (Yeah, you know me!)

3. Find all possible values for the units digit of each person's positive integer.

a. Amy: "When you add 5 to my number, it ends in a 2."
b. Brandon: "When you multiply my number by 3, it ends in a 7."
c. Carmen: "When you multiply my number by 6, it ends in a 4."
d. David: "When you multiply my number by 5, it ends in a 3. Yup."

4. Unlike "base 10", in *mod 10* the only numbers are the remainders when you divide by 10. In mod 10, 6 + 5 = 1 because 1 is the remainder when 6 + 5 is divided by 10.

Answer all these questions in mod 10.

a. $2 + 2 = \boxed{}$
b. $3 \cdot 4 = \boxed{}$
c. $\boxed{} + 5 = 2$
d. $4 \cdot \boxed{} = 2$
e. $5 \cdot \boxed{} = 3$
f. $\boxed{}^4 = 1$

This is sometimes called *modular arithmetic*. Clock arithmetic is mod 12. Four hours from now, it will be four hours later than it is right now.

That last one says box to the fourth power, by the way.

5. Repeat the previous problem, except this time do the arithmetic in *mod 7* instead of mod 10.

Good news: there are only 7 numbers in mod 7. Bad news: in mod 7, every Monday is the same.

6. Go back to the big table that we all filled in together. What patterns do you notice?

Some examples of bad patterns:

"I noticed that most of the values in the table were numbers."

"All the digits in the table could also be found on a computer keyboard."

"Some of the numbers in the table were bigger than others, while others were smaller."

"The leftmost column increased by 1 each time."

"Purple Gingham."

"New Mexico got to have the first two columns because they're in a different time zone."

We've provided some blank space below for you to write your own bad patterns . . .

Neat Stuff

7. Write each fraction as a base-10 decimal.

a. $\frac{1}{5}$

b. $\frac{1}{25}$

c. $\frac{1}{7}$

d. $\frac{3}{7}$

e. $\frac{2}{7}$

f. $\frac{6}{7}$

g. $\frac{1}{13}$

h. $\frac{2}{13}$

8. Write each base-10 fraction as a base-3 decimal. Some of the answers are already given, in which case—awesome!

a. $\frac{1}{13} = 0.\overline{002}_3$

b. $\frac{2}{13}$

c. $\frac{3}{13}$

d. $\frac{9}{13}$

e. $\frac{10}{13}$

f. $\frac{1}{7} = 0.\overline{010212}_3$

g. $\frac{3}{7}$

h. $\frac{9}{7}$

i. $\frac{6}{7}$

9. Write the base-10 decimal expansion of

$$\frac{1}{142857}$$

10. Marvin wonders what kinds of behavior can happen with the base-10 decimal expansion of $\frac{1}{n}$. Be as specific as possible!

11. If $\frac{1}{n}$ terminates in base 10, explain how you could determine the length of the decimal based on n, without doing any long division.

You now know the entire plot of the horrible movie *Terminator 1/4: 0.25 Day*.

12. Robyn wonders what kinds of behavior can happen with the base-3 decimal expansion of $\frac{1}{n}$.

13. We overheard Sara and Joe still yelling about whether or not the number $.99999\ldots$ was equal to 1. Is it? Be convincing.

A little ditty, bout Sara and Joe. Two mathematical kids doin' the best that they know.

14. **a.** Suppose $ab = 0$ in mod 10. What does this tell you about a and b?

b. Suppose $cd = 0$ in mod 7. What does this tell you about c and d?

It tells you that a through d hog the spotlight too much. No love for the middle of the alphabet in algebra.

15. Investigate shuffling decks of cards into three piles instead of two. What are the options? Does it "work" like it does with two piles?

16. **a.** Investigate the base-10 decimal expansions of $\frac{n}{41}$ for different choices of n. What happens?

b. Investigate the *base-3* expansions of $\frac{n}{41}$ for different choices of n. What happens?

The fraction $\frac{n}{41}$ is still in base 10 here, so don't convert 41 to some other number.

17. **a.** Find all positive integers n so that the base-10 decimal expansion of $\frac{1}{n}$ repeats in exactly 4 digits.

b. Find all positive integers n so that the base-3 "decimal" expansion of $\frac{1}{n}$ repeats in exactly 5 digits.

18. Write 223 and 15.125 in base 2. Then write them in base $\sqrt{2}$. How cool is that?!

While this problem is cooler than most math, the Supreme Court recently ruled that math cannot actually be cool.

Tough Stuff

19. Aziz has a cube, and he wants to color its faces with two different colors. How many different colorings are possible? By "different" we mean that you can't make one look like the other through a re-orientation.

20. Barbara has an octahedron, and she wants to color its vertices with two different colors. How many different colorings are possible? By "different" we mean that you can't make one look like the other through a re-orientation.

21. What about edges?

Edges? Edges? We don't need no stinkin' edges!

22. Find all solutions to $x^2 - 6x + 8 = 0$ in mod 105 without use of any technology. There's probably more.

Table for Problem Set 3

This table gives the number of shuffles necessary to restore an n-card deck to its original state using the shuffling style from Problem Set 1.

# cards	# shuffles
4	2
6	4
8	3
10	6
12	10
14	12
16	4
18	8
20	18
22	6
24	11
26	20
28	18
30	28
32	5
34	10

# cards	# shuffles
36	12
38	36
40	12
42	20
44	14
46	12
48	23
50	21
52	8
54	52
56	20
58	18
60	58
62	60
64	6
66	12

Problem Set 3: Suln'hfig

Opener

With an even number of cards, there is a different way to do a "perfect shuffle" by starting from the bottom half instead of the top. For example,

$$123456 \Rightarrow 415263$$

What changes? Determine the number of shuffles needed for different deck sizes.

Today's title is a rejected name of a Sesame Street character. The character's real name is much harder to spell. Wait, no, it's actually the Klingon word for killing someone while shuffling.

Important Stuff

1. What's the difference between *mod 7* and *base 7*? Write a brief explanation (with a numerical example) that a middle-school student could understand.

2. Using the shuffle from *today's opener*, describe the path taken by the top card when you repeatedly shuffle an eight-card deck.

Except for holidays and weekends, we seem to be shuffling at least once in each 24-hour period.

3. These are the *entire tables* for addition and multiplication in mod 7. FINISH THEM!!!

MODALITY! Sub-zero equals six. But only in mod 7. Get it? Whatever.

+	0	1	2	3	4	5	6
0							
1							
2				5			
3							
4							
5			0				
6							5

×	0	1	2	3	4	5	6
0							
1							
2				6			
3	0						
4							
5			3				
6							1

4. Use the tables you built to find all solutions to each equation *in mod 7*. Some equations may have more than one solution, while others may have none.

   a. $5 + a = 4$
   b. $4 \cdot b = 3$
   c. $(5 \cdot c) + 6 = 1$
   d. $d^2 - 4 = 0$

A Higgs boson walks into a church. "What are you doing here?" asks the priest. "You can't have mass without me!" replies the Higgs boson.

5. **a.** Build addition and multiplication tables for mod 10.
b. Solve the four equations from Problem 4 in mod 10.

6. **a.** Are there negative numbers in mod 7? Does any number behave like -1?
b. Are there perfect squares in mod 7? How many?
c. Are there powers of 2 in mod 7? How many?

Sub-zero says hi.

The number 1 is a power of any positive integer b, since $b^0 = 1$.

7. **a.** Gabriel is calculating the powers of 3 in mod 100. Compute the next three entries in the sequence.

$$1, 3, 9, 27, 81, 43, 29 \dots$$

b. Compute the sequence of powers of 3 in mod 3. Uhh.
c. Gail declares that mod 3 wasn't very interesting, and demands that you compute the sequence in mod 7.

DOES NOT COMPUTE . . . OH WAIT IT TOTALLY DOES. Figure out how to do this without calculating $3^7 = 2187$.

Gail = Gabriel + US - rebus!

Neat Stuff

8. Jason handed us a cute blue Post-It that said:

$$10^2 + 11^2 + 110^2 = 111^2.$$

It was both very much like and very much unlike a Smurf.

a. Surely the numbers 10, 11, 110, and 111 in the note are in base 2. Check to see if the statement is true in base 2.
b. Hey wait, maybe those numbers are in base 3. Check to see if the statement is true in base 3.
c. Oh, hm, maybe it was in base 4.
d. Sorry, it was actually in base n. What!

9. **a.** Find all the powers of 3 in mod 9. Oh, that was exciting.
b. Find all the powers of 2 in mod 9.

10. **a.** If $n > 2$, is it possible for *every number* to be a power of 2 in mod n?
b. If $n > 2$, is it possible for *every number except 0* to be a power of 2 in mod n?

Alright, part (b) kind of gives away the answer to part (a) here.

11. Euler conjectured that it takes at least k kth powers to add up to another one. For example, $3^2 + 4^2 = 5^2$ but you need three cubes to add up to another cube. In the 1960s this was finally disproven:

$$133^5 + 110^5 + 84^5 + 27^5 = n^5$$

Without a calculator, and hopefully without multiplying it all out, find the value of n.

12. In the opener, we said you could only do this other shuffle with an even number of cards. We lied. Figure out how to do today's shuffle with an odd number of cards. What do you notice?

We finally found the Higgs boson, so now we can redirect all efforts on finding that darn Waldo.

13. Arielle, Becky, and Chelsea were standing in Tuesday's ridiculous lunch line and had an idea. If any two neighbors switch places, it would create a different arrangement... like

$$\text{ABC} \Rightarrow \text{ACB}$$

They decided to make a big graph of all six ways they could be arranged, and all the connections that could lead from one way to another. Your turn!

Note that only *neighbors* may switch places. Arielle and Chelsea can trade places, but not as the first move.

14. Darren gets in the back of the line behind Arielle, Becky, and Chelsea, and they realize they're going to be stuck making a much larger graph. Good luck!

This graph will contain lots of little copies of the last graph! Neat.

15. If a number can be represented as a repeating decimal in base 10, does it have to be a repeating decimal in every other base? If yes, explain why. If no, are there any particular bases in which it *must* be a repeating decimal?

16. Today is 7/5/12, and $7 + 5 = 12$. Oh snap!

- **a.** How many more times this century will there be a day like this? By *this* we mean the next one is August 4, 2012.
- **b.** How many times will Buck Rogers in the 25th Century see a day like this? You may assume that Buck Rogers arrives on January 1, 2401 and remains alive through the entire century.
- **c.** How can the second answer help you check the first?

And the last one is . . . later than the others?

Wow, that is a really cool answer!

17. Predict the length of the base-10 repeating decimal expansion of $\frac{1}{107}$, then see if you were right.

Tough Stuff

18. Predict the length of the *base-2* repeating decimal expansion of $\frac{1}{107}$, then see if you were right.

19. For even n, the maximum number of perfect shuffles needed to restore a deck with n cards to its original state appears to be $n-2$. Find a rule that tells you when an n-card deck will have the maximum number of necessary perfect shuffles.

A Higgs boson walks into a bar. "Want a drink?" asks the barman. The Higgs boson doesn't reply, because it's a Higgs boson, not a person.

20. *It's p-adic number time!* Every 2-adic positive integer looks like it normally does in base 2, except it has an infinite string of zeros to the left. For example

$$7 = ...000000000000000111.$$

Where he at, where he at . . . p-adic numbers, p-adic numbers, p-adic numbers with a baseball bat!

a. Verify that $7+4=11$ using 2-adic arithmetic.
b. What about subtraction? Try $4-3$, and then try $3-4$.
c. Compute the sum

Oh *man* are you going to have to do a lot of borrowing!

$$...111111111111111. + ...0000000000001.$$

d. Compute this sum using 2-adic arithmetic:

$$1+2+4+8+16+\cdots$$

21. Judy's favorite number is the golden ratio $\phi = (1+\sqrt{5})/2$ and the other Judy loves to do arithmetic in base ϕ. The only problem is that even if the only allowable digits in base ϕ are 0 and 1, not every number has a unique representation. Prove that every number has a unique base-ϕ representation if two consecutive 1s are disallowed.

22. **a.** Find all solutions to $x^2-6x+8=0$ in mod 105 without use of any technology. There's a lot of them.
b. Find all solutions to $x^2-6x+8=0$ in mod 1155 without use of any technology.

Tell you what, we'll give you two solutions: $x=2$ and $x=4$. See, now it's not nearly as tough.

Problem Set 4: Shuf'ling Shuf'ling

Opener

Write down all the powers of 2 in mod 17. Write down all the powers of 2 in mod 13.

Perform Thursday-style shuffles on a 16-card deck, tracking the position of the first card. Do it again for a 12-card deck.

On Monday, perfect shuffles were like the McDLT: the top card stayed on top, and the bottom card stayed on the bottom. Thursday's perfect shuffles were more like the McRib, since they made everything move. Wow, what a horrible analogy.

Important Stuff

1. Complete this table.

n	Powers of 2 in mod n	Cycle Length
7	$1, 2, 4, 1, 2, 4, 1, \ldots$	3
9		
11		
13		
15		
17		
19		
21		
23		
25		
27		

2. If you perform Thursday-style shuffles on a 24-card deck, what positions will the top card take?

MORE shuffling? Oh man, in all possible time spans, we're shuffling.

3. **a.** Kathryn has a deck of 12 cards. Write out the order of her cards after a few of the Monday-type shuffles.

1	2	3	4	5	6	7	8	9	10	11	12

b. Kathy has a deck of 12 cards. Write out the order of her cards after a few of the Monday-type shuffles.

0	1	2	3	4	5	6	7	8	9	10	11

c. Kathi has a deck of 10 cards. Write out the order of her cards after a few of the Thursday-type shuffles.

1	2	3	4	5	6	7	8	9	10

d. Why might you want to number the cards starting from 0 or 1 for a particular type of shuffle? Does anything change with an odd number of cards?

Girl look at those numbers. They work out!

4. What are the powers of 2 in mod 51?

5. Explain why, in the 52-card deck we saw on Monday, the second card in the deck returns to its original position in 8 shuffles.

It's times like you would think a Shufflebot would be useful. Sadly, Shufflebot is only programmed to dance and to apologize.

Neat Stuff

6. Perform Thursday-style shuffles on a 20-card deck and track where each card goes. Complete this table.

Card No.	Positions	Cycle Length	Card No.	Positions	Cycle Length
1			11		
2			12		
3			13		
4	$4, 8, 16, 11, 1, 2, 4, \ldots$	6	14		
5			15		
6			16	$16, 11, 1, 2, 4, 8, 16, \ldots$	6
7			17		
8			18		
9			19		
10			20		

7. All 52 cards from the deck we saw on Monday have a cycle. List all 52 cycles. Maybe there is some way to do it without listing the entire cycle for every single card?

8. Really, aren't you sick of calling these Monday and Thursday shuffles? What should we call them? Best names win!

The worst possible names for these shuffles are SkyBlu and Redfoo.

9. The last problem set asked you to find the perfect squares and powers of 2 in mod 7.

a. You can build a multiplication table to find all the perfect squares in mod 15, but there might be other ways. How many perfect squares are there in mod 15?

b. Refer to Problem 1. How many powers of 2 are there in mod 15?

Compare to the result from Problem 6 from Day 3.

10. Our favorite repeater, $\frac{1}{7}$, can be written as a "decimal" in each base between base 2 and base 10. Find each expansion and see if they have anything in common.

11. Corey, Debbie, and Erik are waiting in line, wondering if they can get to any arrangement through these two rules:

- The person in the back of the group may jump to the front: XYZ $\Rightarrow$ ZXY
- The two people at the front of the group may swap places: XYZ $\Rightarrow$ YXZ

Swaps, swaps, swaps! Swaps swaps swaps!

Kan all six possible arrangements be made? Make a graph illustrating the options.

Sorry for party typo. Or *is* it? Perhaps it completes a phrase. Fun fact: LMFAO won the Kids' Choice award for Favorite Music Group, forcing them to declare their name stood for Loving My Friends And Others.

12. Fred joins the back of the group. Under the same rules, decide whether or not all 24 possible arrangements can be made, and make a graph illustrating the options.

13. So $2^8 = 1$ in mod 51. All this actually proves is that the *first* moving card returns to its original position after 8 shuffles. Complete the proof by showing that every other card also returns to its original position after 8 shuffles.

14. Sometimes while shuffling, the deck completely flips. When this happens, all cards appear in reverse order (except for the end cards when using a Monday-style shuffle). Some people observed that when this happens, that was a halfway point to the shuffling. Is this true? Explain why or why not.

Flipping your deck while shuffling sounds like one of the greatest breakdancing moves ever. Breakdancing hasn't been the same since they cancelled production on Breakin' 3: The Boogaloo Kid.

15. When using Thursday-style shuffles, for what deck sizes does the deck completely flip?

16. Monday's 52-card shuffle didn't have a flipped deck at 4 shuffles, because we would have noticed that. But does anything interesting happen at the 4th shuffle? Look carefully and compare the deck after 4 shuffles to the original deck. Can you explain why this happens?

The observation here may be easier with two decks of cards; one to shuffle, and one to leave in the original setup.

17. Find a mathematical equation that is true in mod 2 and mod 3, but not true in general.

18. Find a mathematical equation that is true in mod 2, mod 3, mod 4, and mod 5, but not true in general.

Psst: you can skip mod 2. Why?

19. Investigate any connection between the number of powers of 2 in prime mods and the number of powers of 2 in composite mods. Look for an explanation or proof of what you find.

20. *It's p-adic number time!* In 3-adic numbers, non-negative integers are written in base 3 with leading zeros:

Aww yeah! The p-adic numbers make as much sense as most LMFAO videos.

$$16 = ...00000000000121.$$

- **a.** Try $16 - 9$. Hey, that wasn't so bad!
- **b.** Try $16 - 17$. Oh dear.
- **c.** What is the value of $1 + 3 + 9 + 27 + 81 + \cdots$?

Tough Stuff

21. The length of the repeating decimal for $\frac{1}{2}$ in base p, where p is prime, is sometimes even and sometimes odd. When? Find a rule and perhaps a proof even?

The first person to find and prove this will receive a champagne shower! Offer expires 7/5/2012.

22. For what primes p is there an even length of the repeating decimal for $\frac{1}{5}$ in base p?

23. For what primes p is there an even length of the repeating decimal for $\frac{1}{10}$ in base p?

24. Determine and prove the Pythagorean Theorem for p-adic numbers, or decide that this problem is completely bogus and there is no such thing.

It's mathy and you know it.

Problem Set 5: Super Bowl

Opener

Complete this table. Don't worry, none of it is in base 2. Long division is fun! While it's a good idea to split the work among one another, please don't use any technology in this work or you may miss some big ideas.

We are the PCMI Shufflin' Crew. Shufflin' on down, doin' it for you.

Jokes so bad we know we're good. Blowin' your mind like we knew we would.

You know we're just shufflin' for fun, struttin' our stuff for everyone.

Fraction	Decimal representation	# of repeating digits
1/3	$0.\overline{333}$	1
1/5	0.2	n/a
1/7	$0.\overline{142857}$	6
1/9		
1/11		
1/13		
1/15		
1/17		
1/19		
1/21		
1/23		
1/25		
1/27		

Important Stuff

1. **a.** When doing long division, how can you tell when a decimal representation is about to terminate?
   b. How can you tell when a decimal representation is about to repeat?

In other words, when do you get to say "I'm done!" with these problems? I suppose you could really say that right now, but that's no fun, is it? Get back to work!!

2. Explain why the decimal representation of $\frac{1}{n}$ can't have more than n repeating digits.

Is there a better upper bound than n digits?

3. Find all solutions to each equation in mod 10.

a. $0x = 1$	**f.** $5x = 1$
b. $1x = 1$	**g.** $6x = 1$
c. $2x = 1$	**h.** $7x = 1$
d. $3x = 1$	**i.** $8x = 1$
e. $4x = 1$	**j.** $9x = 1$

In *mod 10*, the only numbers are 0 through 9. For example, 6 + 5 = 1. Fridge's number changes to 2, but McMahon gets to keep his 9.

4. **a.** Complete this multiplication table for mod 8 arithmetic.

$\times$	0	1	2	3	4	5	6	7
0								
1								
2				6				
3	0							
4								
5								
6			4					
7								1

In *mod 8*, Fridge would be especially unhappy to see his number changed to 0, since he'd be stuck wearing the same number as the punter with the cowbell and Panama hat.

b. How many 1s did you see in your table above?

Don't include the ones on the sidelines.

5. Numbers can multiply with other numbers to make 1! It happens, but not always. Whenever this happens, both numbers are called *units*.

a. If you're working just with integers, what numbers are units?

When working with integers only, 5 is not a unit, since $5 \times \frac{1}{5} = 1$. But . . .

b. If you're working just with rational numbers, what numbers are units?

c. If you're working in mod 10, what numbers are units?

d. List all the units in mod 8.

e. List all the units in mod 15.

The "Shufflin' Crew Band" and "Shufflin' Crew Chorus" do not count as additional units.

6. Pick some more mods. Try to determine rules for what numbers are units in mod m, and how many units there are. Keep picking more mods until you have a feel for it.

"A Feel For Units" was narrowly rejected as the title for Chaka Khan's greatest hit.

Neat Stuff

7. What size decks will get restored to their original order after exactly 10 Thursday-style shuffles (and not in any fewer number of shuffles)?

I'm Samurai Mike I stop'em cold. Part of the defense, big and bold.

I've been jammin' for quite a while, doin' what's right and settin' the style.

Give me a chance, I'll rock you good, nobody messin' in my neighborhood.

(This man went on to coach the 49ers.)

8. **a.** How many units are there in mod 9? Call this number "Bond".

b. Build a multiplication table for mod 9 *but only include the units*. This multiplication table's size will be Bond-by-Bond.

c. Shuffle an 8-card deck using Thursday's shuffle style, and list the positions of all the cards at each shuffle. Look for something interesting!

Fun Fact: Da Bears were *nominated for a Grammy award* for their performance, and they remain the only professional sports team with a Top 41 hit single. (Wait, there's a Top 41 now?)

9. Can a power of 2 be a multiple of 13? Explain.

10. Multiply out these expressions.

$$(2^a - 1)(1 + 2^a + 2^{2a} + 2^{3a} + \cdots + 2^{(b-1)a}) = ?$$
$$(2^b - 1)(1 + 2^b + 2^{2b} + 2^{3b} + \cdots + 2^{(a-1)b}) = ?$$

What does that tell you about $2^n - 1$ when n has factors?

11. Find the three prime factors of $2^{14} - 1$ astoundingly quickly, by hand.

12. That thing this morning. How'd we do that?

How'd we do what? You know what. The thing, with the thing.

13. With one per table, people picked 14 cards today for our magic trick.

a. Why isn't it a good idea to ask the question "What is the probability that at least two groups picked a duplicate card?"

b. If you draw 14 cards from a deck, with replacement, what is the probability that you pick a duplicate card?

What is the probability that the New England Patriots won the 1985 Super Bowl?

14. What size decks will get restored to their original order after exactly 14 Thursday-style shuffles (and not in any fewer number of shuffles)?

I'm mama's boy Otis, one of a kind. The ladies all love me for my body and my mind.

I'm slick on the floor as I can be, but ain't no sucker gonna get past me.

15. Determine all deck sizes that can be restored to their original order in 15 or fewer shuffles, and the specific number of shuffles needed for each.

16. Nicole, Amanda, Peggy, and Shaffiq are waiting in line, wondering if they can get to any of their 24 possible arrangements through these two rules:

- The person in the back of the group may jump to the front: NAPS $\Rightarrow$ SNAP
- The person in the third position may switch with the person in the first position: NAPS $\Rightarrow$ PANS

Can all 24 possible arrangements be made? Make a graph illustrating the options.

Oh, SNAP!

Hey I'm Bowen, from EDC. I write books called CME.

I stay up late most every night, writin' problems that come out right.

I add numbers fast, just like magic, but my hairline is getting tragic.

You all got here on the double, so let's all do the PCMI shuffle . . .

17. *It's p-adic number time!* Here are two interesting 10-adic numbers, and we're only going to show you their last six digits. (There are more digits to the left, feel free to try and figure out what they are.)

$$x = ...109376.$$

$$y = ...890625.$$

a. Calculate $x + y$.
b. Calculate the product xy.
c. Calculate x^2 and y^2.
d. How *crazy* are the p-adic numbers?!

Willie Nelson? Patsy Cline? Aerosmith? Britney Spears? Gnarls Barkley? Eddie? Horse? Madonna is crazy for $U(n)$.

Tough Stuff

18. Let n be an integer. Let $U(n)$ be the set of all deck sizes that are restored to its original order after exactly, and no fewer than, n Thursday-style shuffles. (You can disregard trivial decks with 0 or 1 cards.) In Problem 7, you calculated $U(10)$.

a. Prove that $U(n)$ is never empty: there is always some deck for which n shuffles is the lowest possible number.
b. For which n does $U(n)$ contain only one element?

My name's Darryl, I'm from LA. I work with math most ev'ry day.

I love to laugh, I love to eat, my Mathematica programs can't be beat.

I've taught kids of every age, now I'm in Park City writin' page by page.

I'm not here to fuss or fumble, I'm just here to do the PCMI Shuffle . . .

19. **a.** For $n = 1$ through $n = 7$, find all n-digit numbers who last n digits match the original number. For example, $25^2 = 625$, ending in 25.
b. Find a connection between this and the p-adic numbers, or decide that there is no such connection.

Problem Set 6: Cupid

Pre-Important Stuff

1. **a.** Find the repeating decimal for $\frac{1}{41}$. List all the remainders you encountered during the long division, starting with 1 and 10.

   b. Write $\frac{100}{41}$ as a mixed number.

   c. Find the repeating decimal for $\frac{18}{41}$. List all the remainders you encountered, including 1 and 10.

   d. Find the repeating decimal for $\frac{1}{37}$. List them remainders!

   e. Find the repeating decimal for $\frac{1}{27}$. Coolio.

Wait what, *Pre-Important* Stuff? We're pregaming and it's only 8 am, that's big trouble.

Don't forget tonight is *Pizza and Problem Solving*. You'll have a good time. Or, at a minimum, pizza.

No, it's not Coolio, the guy's name is Cupid. He even named the dance after himself, which seems a little presumptuous.

Opener

Complete this table. Splitting up the work is a great idea, but please do it without fancy spreadsheets or computer programs.

You're going to have to walk it by yourself, walk it by yourself.

n	Powers of 10 in mod n	Cycle Length
41		
37		
3		
7		
9		
11		
13	$1, 10, 9, 12, 3, 4, 1, \ldots$	6
17		
19		
21		
23		
27		
29		

→ Reminder: You don't have to calculate 10^5 to figure out the 4 in this row. Since you already know the previous 3 is equal to 10^4 in mod 13, you can use $3 \times 10 = 30 = 4$ mod 13. It's super helpful! Also, be careful of Cupid's arrows. This arrow points to the right, to the right.

Important Stuff

2. Consider the numbers n in the opener. Find all n among the list that are . . .

a. . . . factors of 9.

b. . . . factors of 99 but not of 9.

c. . . . factors of 999 but not of 9 or 99.

d. . . . factors of 9999 but not of 9, 99, or 999.

e. . . . factors of 99999 but not of 9, 99, 999, or 9999.

f. . . . factors of 999999 but not of . . . alright already.

What's up with that?

Ferris has been absent 9 times. 9 times? NINE TIMES.

I got 99 factored, and 7 ain't one.

Appropriate Beatles song: Number Nine.

Appropriate Nine Inch Nails song: 999999.

Ooooo weeeee, what up with that, what up with that!

3. Calculate each of the following. You may also want to look back at Problem Set 5.

a. $999999 \div 7$
b. $999999 \div 13$
c. $99999 \div 41$
d. $999999 \div 37$

⇒ Only five 9s this time! Cupid's arrow still points to the right, to the right.

4. Complete this table. A number x is a *unit* in mod n if there is a number y such that $xy = 1$. Yesterday we noticed that this is also the list of numbers in mod n that have no common factors with n.

You can say x is *relatively prime* to n, which is totally different than saying that x is *optimus prime*.

n	Units in mod n	# units in mod n
8	$1, 3, 5, 7$	4
15	$1, 2, 4, 7, 8, 11, 13, 14$	8
25		
30		
49	*too many units*	

← Was that just a *Sneakers* reference? I guess it is now! To the left, to the left.

5. How many units are there in mod 105? Counting them all would be a little painful.

6. Karen points out that the list of units in mod 15 contains 2 and 7, and $2 \times 7 = 14$ is also a unit.

a. Solve $2a = 1$ and $7b = 1$ in mod 15.
b. What is the value of $14ab$ in mod 15?
c. Explain why, if x and y are units in mod 15, then xy is also a unit.

7. Kieran points out that the list of units in mod 15 includes powers of 2: $1, 2, 4, 8$.

- **a.** Write a complete list of all the powers of 2 in mod 15. OK!
- **b.** Explain why, if x is a unit in mod 15, then x^2 is also a unit.
- **c.** Same for x^p for any positive integer power p.
- **d.** Why won't there just be billions of units if you can take any unit to *any* power p and make another one?

OK, Cupid? OKCupid's website says "We use math to get you dates"; its founder has a math degree. *The More You Know . . .*

Neat Stuff

8. Donna asks what size decks will get restored to their original order after exactly 10 Thursday-style shuffles (and not in any fewer number of shuffles).

9. Jen asks for what n does the decimal expansion of $\frac{1}{n}$ have an immediate repeating cycle of 10 digits and no fewer?

For example, $\frac{1}{15} = 0.0\overline{6}$ does *not* immediately repeat.

10. Let $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$ be an ordered list in mod 9.

- **a.** Find a number M such that $2M = 1$ in mod 9.
- **b.** Calculate $M \cdot S$ in mod 9. *Keep everything in its original order.*
- **c.** Calculate M^2S in mod 9.
- **d.** Calculate M^kS in mod 9 for all k until something magical happens, then look back to see just how magical it is.

$\Leftarrow$ M^2S can also be used to send pictures to someone's phone. For large k, M^kS indicates that several people enjoyed a meal. To the left, to the left.

11.

- **a.** What are the units in mod 11?
- **b.** Which of the units in mod 11 are perfect squares? Which are not perfect squares?
- **c.** What are the units in mod 13?
- **d.** Which of the units in mod 13 are perfect squares? Which are not perfect squares?

In mod 11, 5 is a perfect square because $4 \cdot 4 = 5$. It's not the same in mod 13!

12.

- **a.** There are 16 units in mod 17. Of them, make a list of the eight that are perfect squares and the eight that are not.
- **b.** If x and y are perfect squares, is xy a perfect square... always? sometimes? never?

c. If x is a perfect square and y isn't, is xy a perfect square... always? sometimes? never?

d. If neither x nor y is a perfect square, is xy a perfect square... always? sometimes? never?

I get stupid. I shoot an arrow like Cupid. I'll use a word that don't mean nothin', like looptid. Hey, that guy named the dance after himself, too!

13. In shuffles, there are values of k for which there is only one deck size that restores in k perfect shuffles and no fewer. Are there values of k for which there is only one denominator n such that $\frac{1}{n}$ has a k-digit repeating decimal?

14. A *repunit* is a number made up of all ones: 11111 is a repunit. Investigate the prime factorization of repunits, and determine the values of k for which the k-digit repunit is prime.

15. Hey, we skipped $n = 15$ in the opener. What's up with that? Well . . .

a. What is the length of the repeating portion of the decimal representation of $\frac{1}{15}$?

b. What is the smallest positive integer n for which $10^n = 1$ in mod 15?

c. Aren't your answers for the previous two problems supposed to match? Figure out what's going on and rectify the situation.

I said, ooooo weeeee, what up with that, what up with that! Ooooo weeeee, what is *up* with that!

Well, looks like we're out of time.

Tough Stuff

16. **a.** Expand $(x-1)(x-2)(x-3)\cdots(x-6)$ in mod 7.

b. Calculate $1^6, 2^6, 3^6, \ldots, 6^6$ in mod 7.

c. For p prime and $x \neq 0$, prove $x^{p-1} = 1$ in mod p.

17. Turns out you can move the top card in a 52-card deck to any position, with a lot less than 52 shuffles. You just need a sequence of both Monday-style and Thursday-style shufflin' shufflin'. Find a method for moving the top card to any desired position in the deck with six or fewer shuffles.

Turns out lots of people get to name dances after themselves, including Dougie, Hammer, Urkel, Freddie, Macarena, Batman, Bartman, Pee-Wee, Ben Richards, and, of course, Carlton.

Problem Set 7: Slurpee Mix

Opener

Behold!

n	$2^n - 1$	Factors of $2^n - 1$ (less than 100)	Deck sizes restored by n out-shuffles	Deck sizes restored by n in-shuffles
1	1	1	2	—-
2				
3	**7**	1, **7**	8	6
4				
5				
6				
7				
8	255	1, 3, 5, 15, 17, 51, 85	18, 52, 86	16, 50, 84
9				
10	1023	1, 3, **11**, 31, 33, 93	12, 34, 94	
11				

Hey, me just met you, and this is crazy, but you got Slurpee, so share it maybe?

Important Stuff

1. Working with your table, fill in a whole lot of *this* table:
`http://www.tinyurl.com/numberofunits`
A number x is a *unit* in mod n if there is a number y such that $xy = 1$. This is also the list of numbers in mod n that have no common factors with n.

Any table caught violating the instructions in the spreadsheet will be forced to listen to the Super Bowl Shuffle on infinite repeat.

2. What do these equations have to do with $\frac{1}{13}$?

$$\begin{aligned} 1 &= 0 \cdot 13 + 1 \\ 10 &= 0 \cdot 13 + 10 \\ 100 &= 7 \cdot 13 + 9 \\ 90 &= 6 \cdot 13 + 12 \\ 120 &= 9 \cdot 13 + \\ &= 2 \cdot 13 + \\ &= \quad \cdot 13 + \end{aligned}$$

Finish the equations, then use the same method to find the decimal expansions of $\frac{2}{13}$ and $\frac{1}{\mathbf{7} \cdot \mathbf{11}}$.

Oops, we didn't finish it all! For a colorful version, see the online class notes. If you're already looking at the online class notes, please stop reading this sentence... now. (Good.)

3. **a.** List the powers of 10 in mod $\mathbf{7 \cdot 11}$:

$$1, 10, \ldots$$

b. How long is the repeating decimal for $\frac{1}{\mathbf{7 \cdot 11}}$?

c. Explain why the length of the repeating decimal for $\frac{1}{n}$ is the same as the length of the cycle of powers of 10 in mod n.

I'm at the Pizza Hut, I'm at the Taco Bell. I'm at the Taco Bell, I'm at the Pizza Hut. I'm at the permutation Pizza Hut and Taco Bell!

4. Complete this table with help from techmology. Woooooww!

n	# units in mod n	$2^{(\text{\# units in mod } n)}$ in mod n	$10^{(\text{\# units in mod } n)}$ in mod n
13			
17			
21			
41			
51			
$\mathbf{7 \cdot 11}$			

What implications does this table have for shuffling cards and repeating decimals?

Yo. Science, what is it all about. Techmology. What is that all about? Is it good, or is it wack?

Neat Stuff

5. Suppose you know that in mod n, there's a number $x \neq 1$ that makes $x^{10} = 1$.

a. Find some other integers k for which you are *completely sure* that $x^k = 1$.

b. Find some integers $k < 10$ for which it is *possible* that $x^k = 1$.

c. Find some integers $k < 10$ for which it is *definitely impossible* that $x^k = 1$.

There are four different **7-11**s in Park City. Plenty of places to get free Slurpees.

6. Suppose you know that in mod n, $ab = 1$ and $cd = 1$.

a. Explain why a, b, c, and d are all units in mod n.

b. Find something that solves this equation:

$$(ac) \cdot (\qquad) = 1$$

This proves that the product of two units is a unit.

c. Find something that solves this equation:

$$(a^k) \cdot (\qquad) = 1$$

This proves that the power of a unit is a unit, and the daughter of a Zappa is also a unit.

7. Let $H = \{1, 2, 3, 4, 5, 6, \mathbf{7}, 8, 9, 10\}$ in mod **11**.

a. Find a number A such that $2A = 1$ in mod **11**.

b. Calculate $A \cdot H$ in mod **11**. *Keep everything in its original order.*

c. Calculate A^2H in mod **11**.

d. Calculate A^kH in mod **11** for all k until something magical happens.

Wait, Problem **7** is about mod **11**? Boom goes the dynamite.

$\Leftarrow$ A^2H indicates that you're at the dentist after too many Slurpees. For large k, A^kH indicates that zombies are nearby.

8. Kathi suggests you recheck your work from Problem 3 on Set 4. Notice anything?

9. **a.** There are 6 units in mod **7**. For each number x in mod **7**, compute x^6 in mod **7**. Cool!

b. There are 4 units in mod 8. For each number x in mod 8, compute x^4 in mod 8. Cool?

Cool. Cool cool cool.

NOT COOL!! Unless you are also drinking a Slurpee, in which case you are automatically cool.

10. **a.** Write out all the units in mod 15.

b. Pick one of them and call it Vo. Now multiply Vo by all the units in mod 15 (including itself). What numbers do you get?

c. If the units are called $u_1, u_2, u_3, \ldots, u_8$ show that

$$u_1 \cdot u_2 \cdot u_3 \cdots u_8 = (Vo \cdot u_1) \cdot (Vo \cdot u_2) \cdot (Vo \cdot u_3) \cdots (Vo \cdot u_8)$$

d. Show that $(Vo)^8 = 1$ in mod 15.

11. The decimal expansion of $\frac{1}{\mathbf{7}}$ is $0.\overline{142857}$. Now, split the repeating digits in half and add them together:

$$142 + 857 = 999.$$

Try this with other fractions $\frac{1}{n}$ that have an even number of repeating digits. Any ideas why this works?

Some examples include $\frac{1}{13}$, $\frac{1}{73}$, $\frac{1}{91}$, and for the adventurous, $\frac{1}{17}$ and $\frac{1}{23}$.

12. One way some people counted up the units in mod 105 was this: *I'll take all 105 numbers, then subtract the number of multiples of 3, then 5, then* ***7****. Oh, wait, then I have to add back in the multiples of 15, 21, and 35. Oh, wait, then I have to subtract the multiples of 105.*

a. Does this work? There are 48 units in mod 105.

b. What's the probability that a number picked between 1 and 105 *isn't* a multiple of 3? 5? **7**?

c. Consider the product $(3-1)(5-1)(\mathbf{7}-1)$. Discus?

From Cookie Monster's "Share It Maybe": Me start to really freak out, please someone call a Girl Scout, don't mean to grumble or grouse, this taking toll on my house . . .

Discus not as good as cookie or Slurpee.

13. Change the argument a little from Problem 10 to show that for any unit in any mod,

$$u^{(\#\text{ units in mod } n)} = 1 \text{ in mod } n$$

14. Watch the 6 of spades as a full 52-card deck undergoes Monday-style shuffles. Before each of the shuffles occurs, make a note of whether it was in the first or second half of the deck. If it is in the first half of the deck, write a 0 in the appropriate spot in the table below. If it is the second half of the deck, write a 1. Cupid says you should do it again with the 10 of diamonds, and there is room for you to pick your own card to try.

See the animation on the "Class Notes" page on the SSTP 2012 Mathforum web site. The spades come first, then hearts, clubs, and diamonds.

To the right, to the right . . . come on, you know the rest! And so does the 10 of diamonds, apparently!

Card	Position just before shuffle #							
	1	2	3	4	5	6	7	8
6♠								
10♢								

a. For each card, compute the base-2 number that corresponds to the eight 0s and 1s you listed.

b. Given a card, is there a way to construct its sequences without watching the shuffles?

These numbers run from 0 to 255.

Since you asked, probably, yes?

15. Carl hands you his favorite multiple of 5, written as an eight-digit base-2 number: $011010??_2$. Oh noes, you can't make out the last two digits. What must those missing digits be for the number to be a multiple of 5?

Last year, Slurpees were also free on **7**/**11**. Slurpee sales that day were 38% *higher* than normal, even though they were giving them away for free.

Tough Stuff

16. Consider different odd primes p and q. The number p may or may not be a perfect square in mod q, and the number q may or may not be a perfect square in mod p. Seek and find a relationship between these two things! Respek.

There is so little respek left in the world, that if you look the word up in the dictionary, you'll find that it has been taken out.

17. Consider different odd primes p and q. The length of the repeating decimal of $\frac{1}{p}$ in base q may be odd or even, and the length of the repeating decimal of $\frac{1}{q}$ in base p may be odd or even. Seek and find a relationship between these two things! Booyakasha.

Problem Set 8: Miscellaneous

Opener

What do these equations have to do with the base 2 "decimal" for $\frac{1}{21}$?

$$
\begin{aligned}
1 &= 0 \cdot 21 + 1 \\
2 &= 0 \cdot 21 + 2 \\
4 &= 0 \cdot 21 + \\
8 &= \quad \cdot 21 + 8 \\
16 &= 0 \cdot 21 + \\
32 &= 1 \cdot 21 + \\
22 &= 1 \cdot 21 +
\end{aligned}
$$

Finish the equations above.

from *Hollywood Shuffle*: Ain't nothin' to it, but to do it! This movie featured Keenen Ivory Wayans as "Jheri Curl" (he also wrote the movie).

Important Stuff

1. Complete this table.

n	"Decimal" for $1/n$ in base 2	# of repeating digits	# of units in mod n
3	$0.\overline{01}$	2	2
5			
7			
9			
11			
13			
15			
17			
19			
21	$0.\overline{000011}$	6	

Using the method from the opener may be helpful to your sanity, but you can also use long division. Split the work, but keep away from techmology or you may miss some big ideas.

The *Ickey Shuffle* is the most famous touchdown dance of all time. One of the two TV announcers from the movie *Cars* did the Ickey Shuffle after winning a NASCAR race. For more information, watch the "Return of the Shirt" episode from *How I Met Your Mother*. It's . . . wait for it . . .

2. **a.** Write out all the units in mod 21.
b. Write out all the powers of 10 in mod 21, starting with 1 and 10.
c. Take all the powers of 10 in mod 21 and multiply them by 2. What happens?
d. Take all the powers of 10 in mod 21 and multiply them by 3. What happens?
e. Take all the powers of 10 in mod 21 and multiply them by 7. What happens?

. . . legendary!!

Hey, you're still in mod 21! There's no such number as 32.

3. Find each decimal expansion in *base 10*. Seek shortcuts to simplify your work!

If you use a calculator, report your answers as repeating decimals instead of rounding them off.

Fraction	Decimal	Fraction	Decimal
1/21	$0.\overline{047619}$	11/21	
2/21		12/21	
3/21		13/21	
4/21		14/21	
5/21		15/21	
6/21		16/21	
7/21		17/21	
8/21		18/21	
9/21		19/21	
10/21		20/21	

Ali: "Now hold it. The *Ali Shuffle* is a dance that will make you scuffle. During the time that I'm doing this shuffle, for a minute, you're going to be confused. You must get in a boxing position, and have a little dance." Howard Cosell – do the voice: "What we've just seen perhaps is the heavyweight champion of the world in what should be his true profession, that of a professional dancer."

4. **a.** How many fractions in the table above are in lowest terms?
b. How many units are there in mod 21?
c. Pick three different units in mod 21. For each, calculate u^{12} in mod 21.

A fraction is in *lowest terms* if it cannot be reduced. 7/11 is in lowest terms, but 6/10 is not.

5. Describe any patterns you notice in the table. What fractions form "cycles" that use the same numbers in the same order? Write out each cycle in order. How long are the cycles?

Coicles? I don't see any coicles here, nyuk nyuk nyuk.

6. Let's try it again, but this time we'll use *base* 2. Find each "decimal" expansion. Seek shortcuts to simplify your work.

Fraction	Base 2 "Decimal"	Fraction	Base 2 "Decimal"
1/21	$0.\overline{000011}$	11/21	
2/21		12/21	
3/21		13/21	
4/21		14/21	
5/21		15/21	
6/21		16/21	
7/21		17/21	
8/21		18/21	
9/21		19/21	
10/21		20/21	

The *Curly Shuffle* is a 7-cycle on the floor: "Woooooooooo woo woo woo, woo woo woo. Woooooooooo woo woo woo, woo woo woo." See also: Angus Young in Let There Be Rock; Homer Simpson in the "Lisa needs braces" episode.

7. Here is a 22-card deck under Monday-style shuffling.

`http://www.tinyurl.com/22cards`

Follow some cards and follow some remainders. How can you use the cards to find the *entire base-2 expansion* for each fraction?

Where? I don't see it. Oh, it's *in* the computer.

How can we be expected to teach children to learn how to read *if they can't even fit inside the building?*

Neat Stuff

8. So, shuffling. Thomas is so good at it that he suggests you get a deck whose size is a multiple of 3 and try triple-out-shuffling and triple-in-shuffling! Cut the cards into three piles then shuffle them together from either the left or the right.

See what you find. We recommend using the same card notation: with out-shuffles, count cards as 0, 1, 2, ... With in-shuffles, count cards as 1, 2, 3, ... There's a lot to find!

Yeah yeah yeah, shake a tail feather baby . . . Yeah yeah yeah, do the *Harlem Shuffle*. Thursday night is karaoke night . . . maybe this time. The director of Harlem Shuffle's music video created *Ren & Stimpy*!

Curly: I'm tryin' to think, but nothin' happens!

9. **a.** If you haven't yet, go back and do Problem 10 from Set 7.

b. The *order* of a unit u in mod n is the smallest power $k > 0$ such that $u^k = 1$ in mod n. Prove that the order of any unit u in mod n must be a factor of the number of units in mod n.

You've got to go back Marty! Back to . . . oh, the past. Bah.

Anarchy in the u^k!

10. **a.** OK, so about that "magic" thing we did back on Monday. How'd we do that?

b. Here's a hint from Sousada: go back and do Problems 14 and 15 on Set 7 if you haven't already.

c. Here's another hint from Soledad: look at the eight numbers in the table for the 6♠ row. Pretend those numbers are written after a "decimal point" and are repeating digits in a base-2 "decimal" expansion. What is the value of this repeating base-2 "decimal"?

You've got to go back Marty! *Shut up.* Oh, and if you see anyone claiming that today is the day from *Back to the Future II*, tell them to shut up, because they've fallen for a bad Photoshop for like the fifth time. The real future date is October 21, 2015, and it predicted all sorts of ridiculous crap like Miami being in the World Series, wall-mounted TVs that would show multiple channels at once, ridiculous numbers of 3D movie sequels, video games you could control with your hands, and video conferencing. Oh.

11. The decimal expansion of $\dfrac{1}{7}$ is $0.\overline{142857}$. Now, split the repeating digits in thirds and add them together:

$$14 + 28 + 57 = 99.$$

Try this with other fractions $\frac{1}{n}$ whose repeating digit lengths are multiples of 3. What's up with that!

12. Suppose a and b are relatively prime. How does the length of the decimal expansion of $\frac{1}{ab}$ compare to the lengths of the decimal expansions of $\frac{1}{a}$ and $\frac{1}{b}$?

13. In a Reader Reflection in the *Mathematics Teacher* (March, 1997), Walt Levisee reports on a nine-year-old student David Cole who conjectured that if the period of the base-10 expansion of $\frac{1}{n}$ is $n - 1$, then n is prime. Prove David's conjecture, and discuss whether it might apply to other bases.

Tough Stuff

14. Prove that if p is an odd prime, there is at least one base $b < p$ in which the expansion of $\frac{1}{p}$ has period $p - 1$. Bonus: determine the number of such bases in terms of p.

Problem Set 9 Handout

Oh my. This is an obnoxious table.

Fraction	Base 2 “Decimal”
1/51	$0.\overline{00000101}$
2/51	
3/51	
4/51	
5/51	
6/51	
7/51	
8/51	
9/51	
10/51	
11/51	
12/51	
13/51	
14/51	
15/51	
16/51	
17/51	
18/51	
19/51	
20/51	
21/51	
22/51	
23/51	
24/51	
25/51	

Fraction	Base 2 “Decimal”
26/51	
27/51	
28/51	
29/51	
30/51	
31/51	
32/51	
33/51	
34/51	
35/51	
36/51	
37/51	
38/51	
39/51	
40/51	
41/51	
42/51	
43/51	
44/51	
45/51	
46/51	
47/51	
48/51	
49/51	
50/51	$0.\overline{11111010}$

Problem Set 9: iPod

Opener

Use this set of equations to find the base-2 "decimal" for $\frac{5}{51}$.

$$\begin{aligned}
5 &= 0 \cdot 51 + 5 \\
&\quad . \\
10 &= 0 \cdot 51 + 10 \\
20 &= 0 \cdot 51 + 20 \\
40 &= 0 \cdot 51 + 40 \\
80 &= 1 \cdot 51 + 29 \\
58 &= 1 \cdot 51 + \\
&= \quad \cdot 51 + \\
&= \quad \cdot 51 + \\
&= \quad \cdot 51 + 5
\end{aligned}$$

It's Friday! Gotta make my mind up, which seat can I take?

Thank goodness there are no giant tables on this problem set. Let's celebrate! As is customary, we gotta get down. See Carl perform as DJ Monochromatic Rectangle, 8 p.m. at Sidecar.

What would William Shatner yell if he was upset with the pedagogical style of a tutoring website? Answer later.

Important Stuff

1. **k.** John demands you write all the units in mod 51.
   a. Write all the powers of 2 in mod 51, starting with 1.
   l. Take all the powers of 2 in mod 51 and multiply them by 2. What happens?
   i. Take all the powers of 2 in mod 51 and multiply them by 5. What happens?
   n. Take all the powers of 2 in mod 51 and multiply them by 17. What happens?

Dave and Brian McKnight both agree that the list of powers ends back at 1.

Like Anthrax, you're caught in a mod! Once you're in a pattern you are stuck. It can't just go bipolar on you and change.

2. **a.** Without converting to a fraction, find a base-10 decimal such that

$$0.\overline{291594} + 0.\overline{mariah} = 1$$

   n. Without converting to a fraction, find a base-2 decimal such that

$$0.\overline{011001} + 0.\overline{rachel} = 1$$

   a. In the obnoxious table, how do the base-2 expansions of $\frac{1}{51}$ and $\frac{50}{51}$ compare?

It's like Mariah's "Get Your Number".

Wait, there's an obnoxious table? Oh no. *NOOOOOO!* Ask Bowen and Darryl for it.

3. Complete the obnoxious table. Be lazy and avoid the use of techmology.

Oh, bother.

4. Describe any patterns you notice in the obnoxious table. What fractions form "cycles" that use the same numbers in the same order? Write out each cycle in order, and find all the cycles.

5. Here is our happy little 52-card deck doing happy little out-shuffles, just like the very first day.

`http://tinyurl.com/pcmi52cards`

Follow some cards. Follow some remainders. Figure out how you can use the cards to find the *entire base-2 expansion* for a fraction in the form $\frac{n}{51}$.

Steven Tyler: "After some long hard thoughts, I've decided it's time to let go of my mistress 'American Idol'. It was over-the-top fun, and I loved every minute of it. Now it's time to bring Rock Back. ERMAHGERD." The guy actually wrote *ERMAHGERD* . . . in all caps . . . in a press release. There's only one word you can say to react to that.

Neat Stuff

6. Let's look at some eight-digit numbers in base 2.

- **a.** What number is 00011001_2 in base 10?
- **s.** One of the fractions in the obnoxious table has a base-2 expansion of $0.\overline{00011001}$. Figure out what fraction it is without looking at the obnoxious table, and without staring at cards like crazy.
- **h.** Start over with 00101101_2 and $0.\overline{00101101}$.
- **t.** Try it again with 11110000_2 and $0.\overline{11110000}$.
- **e.** What are the largest and smallest possible values of an eight-digit number in base 2?
- **n.** Can you explain why all the entries in the obnoxious table turn out to be multiples of [REDACTED]?

Cupid loves you! To the right . . .

Sorry, the multiplier's been redacted, just like last night's version of YMCA. "It's fun to say Y Y Y Y"? Seriously?

7.

- **z.** The base-2 number $110010ab_2$ is a multiple of 5. What are the missing digits?
- **a.** The base-2 number $001101ab_2$ is a multiple of 5. What are the missing digits, and what multiple of 5 is it?
- **c.** Make your own: pick the first six digits of a base-2 number and try to find the missing digits.
- **k.** That thing we did. How'd we do that?

You know, that thing. While cryptic, this still makes more sense than the lyrics to Mmmbop.

8. Based on the card animations, what do you get for the base-2 expansion for $\frac{51}{51}$? Interesting.

9. **j.** Take all the powers of 2 in mod 51 and multiply them by k. That was fun. Whee! It will be helpful, we promise.

Stay. You only hear what you want to.

o. A 52-card deck returns to its original position in 8 out-shuffles. But some cards return sooner. Find an equation that would be true for any card k that returns to its original position after 2 shuffles, then solve it.

Zero bottles of beer on the wall, zero bottles of beer. You take one down and pass it around, 50 more bottles of beer in mod 51.

s. Find an equation that would be true for any card k that returns to its original position after 3 shuffles, then solve it.

h. FOUR!

That's not even a question. But don't worry, be happy.

10. Use ideas from Problem 9 to show that if u is a unit, then u must be part of a cycle that has the same length as the cycle containing 1.

11. A 36-card deck returns to its original position in 12 out-shuffles. Determine all the cards that return sooner, and the cycle length of each.

12. Here are some shuffle animations. Triple shuffles!!! We promise they're cool.

`http://www.tinyurl.com/27cards`

Go figure stuff out. Decimals in base 3, powers of 3 (in what mod?), cycles, magic tricks, all that jazz. What stays the same? What changes?

Watch that wobble, see that wiggle, taste that jiggle. See, aren't you glad now we didn't choose the theme "Every Day We're Jello Pudd-ing?" Actually that sounds pretty awesome.

13. Determine all cycles of cards in the triple shuffle, and the cycle length of each. Given the number of units in the mod, what cycle lengths are possible?

It continues to be the case that you are shuffling in a typical siderial period.

14. **a.** Find a number x so that $x = 1$ in mod 11 and $x = 0$ in mod 13.

b. Find a number y so that $y = 0$ in mod 11 and $y = 1$ in mod 13.

c. Find a number z so that $z = 5$ in mod 11 and $z = 6$ in mod 13.

KHAAAAAANNNNNNNN!

15. Find a number M so that $M = 2$ in mod 3, $M = 3$ in mod 5, $M = 4$ in mod 7, and $M = 5$ in mod 11. Try this without techmology and with as little guesswork as possible!

Will computahs ever be able to work out what 999999999999999999 9999999999999999999 multiplied by 9999999 ... yo I haven't finished you don't know what I was going to say ... 99899999999.889988999 9999999999 is?

16. Look back at the table from Set 8 you made for the fractions from $\frac{1}{21}$ through $\frac{20}{21}$. All the digits from 0 through 9 are used, and there are 120 total digits in the repeating parts of the expansions.

a. Which digit is used most frequently in the decimal part of the expansions? Which digit is used least frequently?

b. Does this happen in other mods? Other bases?? So cool.

Okay, it's still not cool, but cool by math standards. Not cool like Kool Moe Dee cool, but hey.

17. Figure out a way to shuffle cards so that you could use the cards' positions after each shuffle to read off base-10 decimal expansions of fractions.

18. Nine-year-old David Cole conjectured that if the period of the base-10 expansion of $\frac{1}{n}$ is $n-1$, then n is prime. Using the concept of units, prove David's conjecture, and discuss whether it might apply to other bases.

Tough Stuff

19. Prove that there are infinitely many primes p for which the decimal representation of $\frac{1}{p}$ has cycle length $p-1$.

According to Carol, the group is "going to do Bowen at 3:15" today. That's scary. Anyway, it's a brief tour of the *CME Project* curriculum series and some ideas about specific topics like graphing and factoring.

20. The Fibonacci numbers start with $F_0 = 0, F_1 = 1, F_2 = 1$, etc:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, \dots$$

Show that if m is a factor of n, then F_m is a factor of F_n. For example, $F_7 = 13$ is a factor of $F_{14} = 377$.

21. Show that if F is a Fibonacci number greater than 2, then neither $F^2 - 1$ nor $F^2 + 1$ is prime.

22. Take a grid of circles, 12-by-21, and color each circle with one of four colors. Either find a coloring that does not produce a monochromatic rectangle – a rectangle with all four corners the same color – or prove that such a coloring is impossible.

Once again, Carl plays tonight at 8 at Sidecar! Be there or be rectangular.

Problem Set 10: Under the $\mathbb{Z}$

Opener

You have four cards, arranged this way: 1234. You can perform any number of in-shuffles and out-shuffles on them, in any order. Can you get to all possible arrangements of the four cards? If so, show how. If not, explain why not.

Just look at the cards around you, right behind the math-camp door. Such wonderful mods surround you, what more is you lookin' for!

Important Stuff

1. Laura notices in the opener that if you know the first two of the four cards, you can determine the order of the last two cards.

The weird $\mathbb{Z}$ stands for the integers. It comes from the German word "zahlen", meaning "weird-looking Z", and was first used by Evil Emperor Zurg.

- **a.** If the first two cards are 13, what is the order of the last two cards?
- **b.** If the first two cards are 34, what is the order of the last two cards?
- **c.** Can the first two cards ever be 14?
- **d.** What is going on?

Don't forget, this Thursday night is the "Enchantment Under The $\mathbb{Z}$" dance! Be there, or be rectangular.

2. **a.** Follow the 6 of spades and the 8 of diamonds in a regular deck of cards through the sequence of out-shuffles.

`http://tinyurl.com/6spades8diamonds`

What do you notice?

- **b.** Compare the base-2 "decimal" expansions of $\frac{5}{51}$ and $\frac{46}{51}$. What do you notice?
- **c.** Find other pairs of cards with the same behavior.

In $\mathbb{Z}$ all the numbers happy, they glad 'cause it's normal math. Numbers in the mod ain't happy, they stuck in a looping path.

3. You have six cards, arranged this way: 123321. Cards with the same number are identical. You can perform any number of in-shuffles and out-shuffles, in any order. Can you get the first three cards to take on all six possible arrangements of 123? If so, show how. If not, explain why not.

One of the other arrangements is 231132. The first half tells you what the last half has to be, so you could just read it as 231.

4. Build an addition table for $\mathbb{Z}_6$. Wait what? Oh, that's just mod 6. Also, please would you kindly build a multiplication table for $\mathbb{Z}_9$.

Look, it's Andrew Ryan's favorite phrase. (Obscure, but also under the sea!)

$(\mathbb{Z}_6, +)$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1					
2	2					
3	3					
4	4					
5	5					

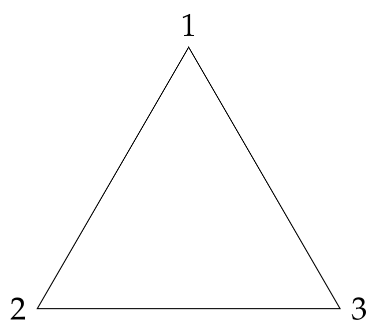
$(\mathbb{Z}_9, \times)$	0	1	2	3	4	5	6	7	8
0		0							
1	0	1	2	3	4	5	6	7	8
2		2							
3		3							
4		4							
5		5							
6		6							
7		7							
8		8							

5. Build a multiplication table for mod 9 that includes only its units. This table's size will be Bond-by-Bond. The set of units of $\mathbb{Z}_n$ is called $\mathbb{U}_n$.

A unit is a lot like a mushroom . . . no!!! A number u is a *unit* if there is a number v that solves $uv = 1$. The multiplication table should help, but we've also found another rule for deciding if a number in a mod was a unit.

If you're using the mathematical practices properly, you can use Problem 5 to model $\mathbb{U}_n$! A joke only teachers could love.

6. Here's an equilateral triangle:



Suppose you can perform any number and sequence of reflections or rotations, as long as you leave one of the corners of the triangle pointing up as shown.

a. Draw all six possible configurations of the triangle.

b. Build an operation table for transforming the equilateral triangle, where the operation is "then". For example, you could rotate the triangle 120 degrees counterclockwise, then reflect the triangle across its vertical line of symmetry. This combination of moves is equivalent to what single move? Do this to complete the table.

This table's size will also be Bond-by-Bond. One of the six options is "do nothing", leaving the triangle in its present orientation.

7. **a.** What is the *identity* for addition?

b. What is the identity for multiplication?

c. Why isn't 0 the identity for multiplication?

d. What is the identity for triangle transformation?

8. The *cycle length* of an element is the number of times you have to repeat its operation to get back to the identity. For example, in $\mathbb{Z}_6$ under addition, the cycle length of 4 is 3:

$$0 + 4 = 4 \text{ then } 4 + 4 = 2 \text{ and then } 2 + 4 = 0$$

a. Find the cycle length for all elements of $\mathbb{Z}_6$ under addition. One of the cycle lengths is 1!

b. Find the cycle length for all elements of $\mathbb{U}_9$ under multiplication. Notice anything interesting?

c. Find the cycle length for all elements of the triangle transformations. Notice anything interesting?

The cycle length of adding 2 in $\mathbb{Z}_{10}$ is 5. The cycle length of multiplying by 2 in $\mathbb{Z}_{51}$ is 8. The cycle length of Dory is about 10 seconds.

And then? *No and then!* And then??

So, it's 1 factorial, or just 1?

I'm crazy for $\mathbb{U}_n$!

I noticed that Triangle Man hates Person Man. They might have a fight.

Neat Stuff

9. Go back to the equilateral triangle. Suppose you're only allowed 120-degree counterclockwise rotations and reflections across the vertical line of symmetry. Can you get to all six possible configurations using only these moves? If so, show how. If not, explain why not.

What is a French chef's favorite statistical distribution? *Le Poisson!* Hee hee hee, haw haw haw!

10. Are in-shuffling and out-shuffling commutative? Explain.

11. Are the six triangle transformations commutative? Explain.

12. Try Problem 6 again with a tetrahedron. You'll have to figure out how many possible configurations there are, then build the operation table.

13. If x is an element of $\mathbb{Z}_n$, show that there must *either* be an element y that solves $xy = 1$, *or* an nonzero element z that solves $xz = 0$, but not both.

We got the spirit, you got to hear it, under the $\mathbb{Z}$.

The two is a shoe, the three is a tree.

The four is a door, the five is a hive.

The six is a chick, the seven's some surfin' guy . . . *yeah*

The eight is a skate, the nine is a sign.

And oh that zero blow!

14. **a.** Find all solutions to $x^2 = 1$ in $\mathbb{Z}_{105}$. There are a lot of them, and busting 105 into little pieces may help.

b. Use completing the square (!) to find all solutions to $x^2 + 24 = 10x$ in $\mathbb{Z}_{105}$.

Near a beach, there was a dangerous tree with a beehive in it. A surfer rode a big wave all the way onto land and crashed into the tree. The bees got mad and chased him! He had to dive back into the water to escape from the bees.

Tree $\times$ hive $\times$ surfin' equals wanna dive!

15. We now know that 52 shuffles will restore a deck of 52 cards using in-shuffles. This is true because $2^{52} = 1$ in mod 53. Suppose you wanted to prove this to a friend, but you only have a simple calculator that can't calculate 2^{52} exactly. Find a way to calculate 2^{52} in mod 53 using as few operations as possible and just a simple calculator. What other powers of 2 would you need to calculate to ensure that it takes 52 shuffles to restore the deck and not some smaller number?

16. **a.** Without a calculator, determine the number of in-shuffles it will take to restore a "double deck" (104 cards) to its original state.

b. How many out-shuffles will it take?

What mod is it for each of these questions?

Tough Stuff

17. Without a calculator, determine the number of in-shuffles it will take to restore a deck of 20,000 cards to its original state.

Oh man. 20,000 cards under the $\mathbb{Z}$? Somebody call Nemo.

18. Jay says there's this box. It's got integer dimensions, like 6-by-8-by-10 but not.

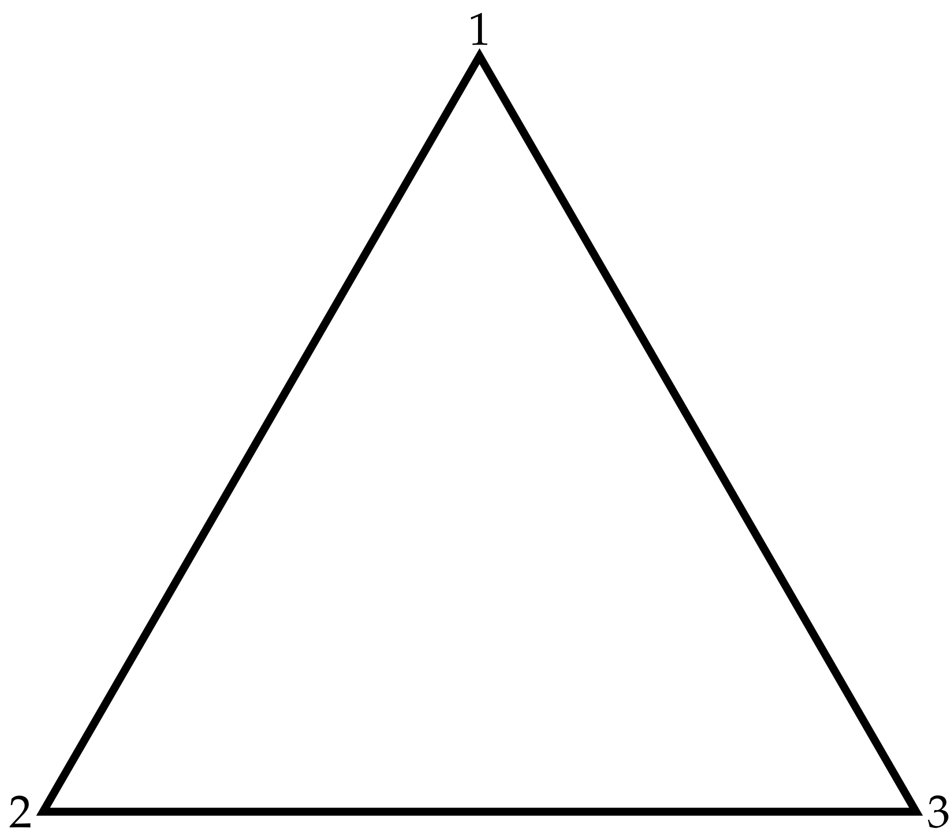
a. All three of the diagonals on the faces of the box also have integer length. Find a possible set of dimensions for the box, or prove that no such box can exist.

b. Additionally, the space diagonal (from one corner of the box to the other corner in eye-popping 3D) also has integer length. Find a possible set of dimensions for the box, or prove that no such box can exist.

Under the Sea 3D is now playing at a theater near you! Well, if by *near* you mean the Omnitheater at the Science Museum of Minnesota, which is literally the nearest theater playing that. While you're there, visit the new *Math Moves!* exhibit with this totally sweet perspective-drawing thing.

Problem Set 11 Handout

Here's a spot to put your triangle!



The numbers indicate the starting position of the triangle.

n	# of units in mod n	Cycle length of	
		2^k in mod n	10^k in mod n
3	2	2	1
7	6	3	6
9	6	6	1
11	10	10	2
13	12	12	6
17	16	8	16
19	18	18	18
21	12	6	6
23	22	11	22
27	18	18	3
29	28	28	28
31	30	5	15
33	20	10	2
37	36	36	3
39	24	12	6
41	40	20	5
43	42	14	21
47	46	23	46
49	42	21	42
51	32	8	16
53	52	52	13
57	36	18	18
59	58	58	58
61	60	60	60
63	36	6	6
67	66	66	33
69	44	22	22
71	70	35	35
73	72	9	8
77	60	30	6
79	78	39	13
81	54	54	9
83	82	82	41
87	56	28	28
89	88	11	44
91	72	12	6

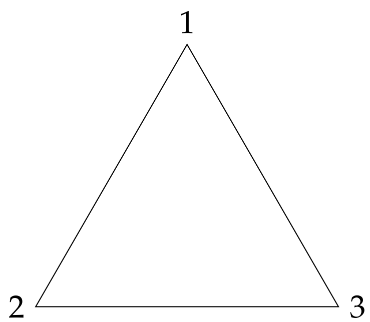
n	# of units in mod n	Cycle length of	
		2^k in mod n	10^k in mod n
93	60	10	15
97	96	48	96
99	60	30	2
101	100	100	4
103	102	51	34
107	106	106	53
109	108	36	108
111	72	36	3
113	112	28	112
117	72	12	6
119	96	24	48
121	110	110	22
123	80	20	5
127	126	7	42
129	84	14	21
131	130	130	130
133	108	18	18
137	136	68	8
139	138	138	46
141	92	46	46
143	120	60	6
147	84	42	42
149	148	148	148
151	150	15	75
153	96	24	16
157	156	52	78
159	104	52	13
161	132	33	66
163	162	162	81
167	166	83	166
169	156	156	78
171	108	18	18
173	172	172	43
177	116	58	58
179	178	178	178
181	180	180	180

Problem Set 11: 𝕌 Can't Touch This

Ma, ma, ma, ma math it hits me so hard
Makes me say "Er, mah gerd"
Thank 𝕌 for asking me
To take triangles and shuffle their feet

Opener

Let's revisit the transformations of the equilateral triangle from the last problem set. But this time, we'll use wacky *permutation notation* to describe the six transformations you can perform. Here are the six transformations.



Permutation	Transformation on the triangle
$()$	Do nothing
$(1\ 2\ 3)$	
$(1\ 3\ 2)$	
$(1\ 2)$	
$(1\ 3)$	
$(2\ 3)$	

$(1\ 2\ 3)$ means 1 goes to 2, 2 goes to 3, 3 goes to 1.

$(1\ 3)$ means 1 goes to 3, 3 goes to 1.

Now complete this operation table, where the operation is "then". For any cell in the table, perform the transformation that labels the row first, then the transformation that labels the column. Write the transformation that is equivalent to the combination of those two transformations.

"then"	$()$	$(1\ 2\ 3)$	$(1\ 3\ 2)$	$(1\ 2)$	$(1\ 3)$	$(2\ 3)$
$()$						
$(1\ 2\ 3)$						
$(1\ 3\ 2)$						
$(1\ 2)$						
$(1\ 3)$						
$(2\ 3)$						

Give me a square or rectangle
Findin' symmetries at every angle
2 . . . Legit! Find them all or
𝕌 might as well quit

That's word because 𝕌 know . . .

Important Stuff

1. Marina hands you this 8-card deck:

0	1	2	3	4	5	6	7

a. Using this deck, explain why an out-shuffle can be represented by this permutation notation:

$$(1\ 2\ 4)(3\ 6\ 5)$$

Stop! Shuffle time. Hey wait, why isn't there a 0 or a 7 in this notation?

b. Write permutation notation for an in-shuffle using this deck.

c. Use the permutation notation to determine the number of out-shuffles needed to restore an 8-card deck to its original position, and the number of in-shuffles needed.

2. Build an addition table for $\mathbb{Z}_4$. Oh, that's just mod 4. Also, build a multiplication table for $\mathbb{U}_5$. Remember $\mathbb{U}_5$ contains all the units in mod 5, which are 1, 2, 4 and 3. What's with the weird order? That's word.

What's a $\mathbb{Z}_4$? For ending the alphabet, silly. $\mathbb{U}_5$ at that table had better get back to work.

$(\mathbb{Z}_4, +)$	0	1	2	3
0				
1				
2				
3				

$(\mathbb{U}_5, \times)$	1	2	4	3
1				
2				
4				
3				

3. Leah says that $\mathbb{Z}_4$ is "just like" $\mathbb{U}_5$. Say *what*??

I'm just like $\mathbb{U}$! I'm just like $\mathbb{U}$ . . .

4. In the previous problem set, you built the addition table for $\mathbb{Z}_6$ and the multiplication table for $\mathbb{U}_9$. Can you make those two tables match by pairing the numbers up in some way? Explain.

5. Now compare the addition table for $\mathbb{Z}_6$ with the operation table for the triangle transformations. Can you make those two tables match by pairing the entries in some way? Explain!!!

Use cycle lengths to help you! The *cycle length* is the number of times an element's operation repeats until the identity appears. For addition, solve $x + x + \cdots + x = 0$ and figuring out how many x's are needed. For multiplication . . . and for triangle transformation . . .

6. **a.** Find the cycle length for each element of $\mathbb{Z}_4$ under addition.

b. Build a multiplication table for $\mathbb{U}_8$, then find the cycle length for each element under multiplication.

7. Can you match the addition table for $\mathbb{Z}_4$ and the multiplication table for $\mathbb{U}_8$ by pairing numbers up in some way?

8. Look at the table on today's handout. Find an important relationship between the number of units in mod n and the cycle length of 2^k in mod n that is consistently true. Do it again between the number of units in mod n and the cycle length of 10^k in mod n.

Are $\mathbb{U}$ glad $\mathbb{U}$ didn't have to fill this out?

Here are some unimportant relationships between the columns:

"The columns all contain numbers."

"The columns all contain positive numbers."

"The columns all contain one-, two-, and three-digit numbers."

"None of the entries in any column is the number 8675309."

Neat Stuff

9. Write the 52-card out-shuffle using permutation notation, and use it to explain why the deck is restored in eight out-shuffles.

10. **a.** In $\mathbb{U}_9$ there are six numbers: 1, 2, 4, 5, 7, 8. The number 2 is called a *generator* of $\mathbb{U}_9$ because every number is a power of 2: $1, 2, 4, 8, 7, 5, 1, \ldots$. Which other numbers in $\mathbb{U}_9$ are generators?

b. What are the generators in $\mathbb{Z}_6$ under addition? Instead of using powers (repeated multiplication) of numbers, you will need to use repeated addition.

c. How can generators help you match the tables for $(\mathbb{U}_9, \times)$ and $(\mathbb{Z}_6, +)$?

It tours around $\mathbb{U}_9$, from 1
then all the way
It's 2, go 2, generate it now
2
That's all there is to say

11. What are the generators in $\mathbb{Z}_7$ under addition? In $\mathbb{Z}_8$? Neat.

Of course it's neat, this is Neat Stuff.

12. Describe some conditions under which you can say that the tables for two operations can *definitely not* be matched. Find more than one condition!

I told you, homeboy, $\mathbb{U}$ can't match this!

13. Find all $n \neq 8$ so that the table for $\mathbb{U}_n$ under multiplication can match the table for $\mathbb{U}_8$ under multiplication.

14. Make a table of the number of generators of $\mathbb{U}_n$ for different n. What patterns do you notice?

I love $\mathbb{U}$, $\mathbb{U}$ love me. Oh no.

15. **a.** Find the cycle lengths of 3 and 5 under multiplication in $\mathbb{U}_{13}$ and explain why each is *not* a generator.

b. What are the generators of $\mathbb{U}_{13}$?

Never mind, I'll find someone like $\mathbb{U}$ . . .

c. So 3 and 5 are not generators. However, Ruth says that any number in $\mathbb{U}_{13}$ can be written as $3^a \cdot 5^b$ for some a and b. Show that she's right!

How many elements are in $\mathbb{U}_{13}$? How long are the cycles of 3 and 5? Hmm.

16. Show that no single transformation is a generator for the six triangle transformations, but that it is possible to choose two transformations that generate all six transformations together.

Yo, sound the bell, school is in, sucka!

17. Here are six functions.

- $m(x) = 1 - \dfrac{1}{1-x}$
- $o(x) = 1 - x$
- $n(x) = 1 - \dfrac{1}{x}$
- $i(x) = x$
- $c(x) = \dfrac{1}{x}$
- $a(x) = \dfrac{1}{1-x}$

a. Build an operation table for working with these six functions, where the operation is "composition". For example, $n(o(x)) = m(x)$.

b. Which, if any, other tables can match this table by pairing the entries in some way?

Yummy: $n \circ o = m$! It's too bad Hammer never had his own cereal. He did get a cartoon show, though.

18. **a.** Find the number of out-shuffles that it will take to restore a deck of 90 cards. Do this without a calculator and using the most efficient method possible, given what you learned in Problem 8.

b. Find the number of repeating digits in the decimal expansion of 1/73 without long division or a calculator.

Stop! Shuffle time.

$\mathbb{U}$ might be more efficient if $\mathbb{U}$ look back at Problem 15 from Problem Set 10.

I guess the change in my pocket wasn't enough, I'm like, forget $\mathbb{U}$.

Tough Stuff

19. If p is prime, prove that every $\mathbb{U}_p$ has at least one generator.

20. If p is prime, prove that every $\mathbb{U}_p$ has exactly ____________ generators. Hm, looks like we left that spot blank.

Don't worry, tomorrow's problem set is *not* titled "We $\mathbb{R}$ Who We $\mathbb{R}$".

21. Find all *composite* n for which $\mathbb{U}_n$ has generators.

Who are $\mathbb{U}$? Who who, who who?

22. If p is prime, find some general conditions under which the number 2 is *definitely* or *definitely not* a generator in $\mathbb{U}_p$.

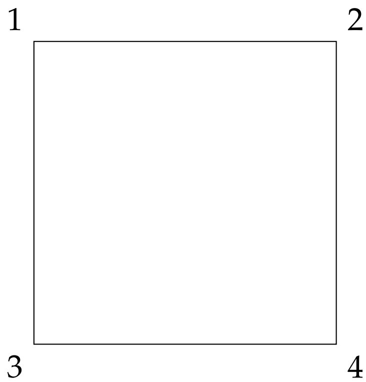
If you missed any of today's comments, look them up on $\mathbb{U}_{\text{tube}}$.

Problem Set 12: Like a d6

Opener

It's squaresville today. What are the *eight* transformations that you can perform on the square below so that it still fits in this space?

This episode of Square One Television is brought to you by the identity letter e, and by the number e.



Permutation	Transformation on the square
$()$	Do nothing
$(1\ 2)(3\ 4)$	Reflection across vertical axis

Cut out your own square and label its corners on both sides as shown in the diagram. A special prize will be given to the most beautiful square!

Poppin' yogurts in Grub Steak, makes us snicker

When we think about some math, it gets twittered

Sippin' water while we learn . . . some math tricks

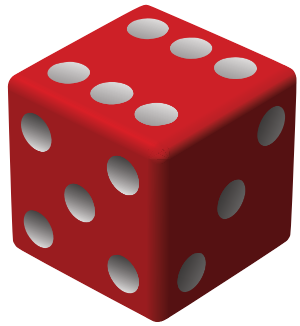
Now I'm countin' symmetries of a d6 . . .

Now complete this operation table, where the operation is "then". For any cell in the table, perform the transformation that labels the row first, then the transformation that labels the column. Write the transformation that is equivalent to the combination of those two transformations.

"then"	$()$							
$()$								

Important Stuff

1. Here is a picture of a d6, with the 6 facing up.



Like a d6, like a d6, now now now now now now I'm feeling like calling it a number cube. Wait, that's probably not how it goes.

   a. Describe all the ways you could move the d6 so that it keeps the 6 on top and still occupies the same space.
   b. How many transformations keep the 6 on top?
   c. How many orientations does the d6 have in total? You no longer have to keep the 6 on top.

We keep it real here at PCMI. All transformations should actually be performable, which rules out something you could do with a square or triangle. Like the square and triangle, the d6 has to end up in the same position, but the orientation could be totally different.

2. Build an operation table for your transformations from Problem 1a, where the operation is "then". You can use any notation that you like. How big should this table be? Where have you seen a similar operation table?

3. Look back at Problem 3 on Set 10. Six cards were arranged this way: 123321. Cards with the same number were identical. Look for the graph on today's handout. Each arrow in the diagram shows how you can get from one arrangement to another using a shuffle. Write an "I" next to each arrow corresponding to an in-shuffle and an "O" next to each arrow corresponding to an out-shuffle.

The "out shuffle" is the Monday shuffle with the stuck cards, and the "in shuffle" is the Thursday shuffle.

4. Laurie, Marian, and Nadine are waiting in line, wondering if they can get to any arrangement through these two rules:
   - The person in the back of the group may jump to the front: LMN $\Rightarrow$ NLM
   - The two people at the back of the group may swap places: LMN $\Rightarrow$ LNM

   Can all six possible arrangements be made? Make or find a graph illustrating the options.

The set of all arrangements of LMN is called the Samsung Galaxy S3, or perhaps just S_3.

Swaps, swaps, swaps! Swaps swaps swaps!

5. Look back to your operation table from the Opener from Set 11.

a. What is the "identity" in this operation table?

b. Which transformations have inverses?

An element with an inverse has been called a *unit*.

c. For each transformation, how many times do you have to perform it to restore the equilateral triangle to its original state?

d. Complete this sentence: "The cycle length of each transformation __________ the total number of transformations."

Bazinga?

6. Titin is waiting in line while holding an equilateral triangle, wondering if she can get to any of the six orientations through these two transformations:

... as you do. It's probably happened a few times here, actually.

- A rotation: $(1\ 2\ 3)$
- A reflection: $(2\ 3)$

Can all six possible orientations be achieved? Make or find a graph illustrating the options.

7. Two groups are called *isomorphic* if there is a correspondence between them that matches their operations completely. Describe at least three isomorphisms you have found in this course so far, and at least two *non*-isomorphisms.

Drink it up, yeah, drink it up When $(U_5, \times)$ around me it be acting like $(Z_4, +)$

Neat Stuff

8. The group S_4 is the set of all possible permutations acting on the numbers 1 through 4, where the operation between the permutations is "then".

a. How many elements are in S_4? Explain how you know.

b. Decide whether or not S_4 is isomorphic to the group of transformations of the square from the opener. What what!

9. A d4 is a tetrahedral die. Picture its faces numbered 1 through 4. As a shorthand, 4 will be represented by the word "fruit".

Worst shorthand ever?

a. Describe all the ways you could move the d4 so that it keeps the fruit on the bottom and still occupies the same space.

Occupy d4!

b. How many transformations keep the fruit on the bottom?

c. How many orientations does the d4 have in total? You no longer have to keep the fruit on the bottom.

Poppin' yogurts in Grub Steak . . .

10. A d8 is an octahedron. How many orientations does it have?

You shoulda had a d8!

11. How many orientations are there if your die is

a. . . . like a d12?

b. . . . like a d20?

c. . . . like a d10?

In the basement rollin' dice, I'm a wizard

When we play we think we fight giant lizards

Now now now now now now don't want my elf to die, roll a d20.

12. An element of a group is a *generator* if repeated operation of that element takes you through every element of the group.

a. Find all the generators for $(\mathbb{Z}_{12}, +)$ or explain why there aren't any.

b. Find all the generators for $(\mathbb{U}_{12}, \times)$ or explain why there aren't any.

c. Find a generator for S_3 or explain why there isn't one.

13. Sometimes $(\mathbb{U}_n, \times)$ has a generator, and sometimes it don't.

Sometimes you feel like a generator, sometimes you don't.

a. Under what conditions will $\mathbb{U}_n$ have a generator?

b. In terms of n, *how many generators* are there?

14. A d6 has the numbers 1 through 6 on it. How many *different* ways are there to put the numbers on a d6? By *different* we mean that there is no transformation taking one arrangement to another.

Roll your own d6, just not in Vegas.

Tough Stuff

15. **a.** Is it possible to generate S_4 using only in- and out-shuffles on cards numbered 12344321?

b. Is it possible to generate S_5 using only in- and out-shuffles on cards numbered 1234554321?

Problem Set 13: The Vertex Of Glory

Opener

On Monday, you made a list of all possible arrangements of the cards 1234 with an unlimited supply of in- and out-shuffles. One side of today's handout contains these eight arrangements. Draw arrows connecting the arrangements. Label each arrow with "I" or "O" to indicate whether the an in- or out-shuffle connects the two arrangements.

An *out-shuffle* keeps the top and bottom cards stationary, while an *in-shuffle* moves everything. I'm your biggest fan, I'm shufflin' until you love me.

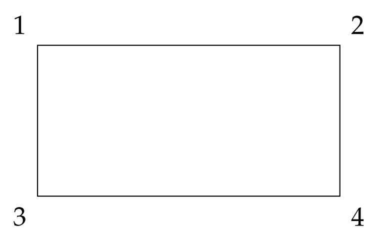
Important Stuff

1. Draw the eight orientations of the square from Set 12's opener. Use arrows in one color to connect two orientations if the $(2\ 3)$ transformation takes one orientation to the other. Use arrows in a different color to connect two orientations if the $(1\ 2\ 4\ 3)$ transformation takes one orientation to another. Don't draw arrows for other transformations. Notice anything?

No matter black, white, or beige. Don't be a drag. You're on the right track, baby.

2. Use your two diagrams to argue that the group generated by in- and out-shuffles on 4 cards is isomorphic to D_4, the group of symmetries of the square.

3. Here's a rectangle:



a. What are the four transformations you can perform on the rectangle so that it still fits in its space?

b. Complete an operation table for the rectangle, where the operation is "then".

c. Find another group you've worked with this week that also has four elements, but is definitely *not* isomorphic to it. Explain how you know they're not isomorphic.

d. Find another group you've worked with this week that has four elements and *is* isomorphic to the rectangle group.

Balderdash time! Six of the seven definitions below are false. Which is the right one?

Alex says that the D in D_4 stands for *Diophanteen*, the latest advance in hair-care technology.

Christina says that the D in D_4 stands for *denom-nomnator*, which is what happens when you can completely cancel out bottom of a fraction.

Dominic says that the D in D_4 stands for *Dominic*, duh.

Eric says that the D in D_4 stands for *de Bruin cycle*, which is how often UCLA wins football games.

Kaelin says that the D in D_4 stands for *de Mauve's Theorem*, which describes when mathematics can and can't be purple.

Gregory says that the D in D_4 stands for *Deerichlay*, which is a fancy wine from the French region of Bourbaki.

Rachell says that the D in D_4 stands for *dihedral*, meaning "two-sided".

4. **a.** A *d4* is a tetrahedral die. How many faces does a d4 have? Alright, not our most brilliant problem ever.
   b. How many different orientations (aka symmetries) does a d4 have?

The d in d4 stands for *die*, meaning "die".

5. How many edges does a d4 have? Use this to verify the number of orientations you found in Problem 4.

6. Determine the number of different orientations for a d6. Find the number of different orientations separately using faces, then using edges, then using vertices.

I want your $\mathbb{Z}_{12}$ and I want your mod 10, you and me could write a bad conjecture.

7. Determine the number of different orientations for a d8. Find those different ways! Sweet.

8. ... for a d12.

9. ... for a d20.

Review Your Stuff

10. We traditionally set aside part of the last problem set for review. Work as a group at your table to write **one** review question for tomorrow's problem set. Spend **at most 15 minutes** on this. Make sure your question is something that $\boxed{*\textbf{everyone}*}$ at your table can do, and that you expect $\boxed{*\textbf{everyone}*}$ in the class to be able to do. Problems that connect different ideas we've visited are especially welcome. We reserve the right to use, not use, or edit your questions, depending on how much other material we write, the color of the paper on which you submit your question, your group's ability to write a good joke, and hundreds of other factors.

Is there really such a thing as a "self-reflective process of discovery"? Yes, there really is! Don't believe us? Ask Google, and put quotes around it.

There ain't no reason A and B should be alone

Today, yeah baby, today, yeah baby

I'm on the vertex of glory

And I'm hanging on a corner with you

I'm on the vertex, the vertex, the vertex, the vertex, the vertex, the vertex, the vertex

I'm on the vertex of glory

And I'm hanging on a corner with you! I'm on the corner with you!

Stupid Stuff

11. Draw a table.

12. Solve for x:

$$(P^3x + P^2x + M^4ah)^4 = \text{ can't read my } + x$$

Neat Stuff

13. The rectangle transformation group is usually called D_2, the triangle transformation group is D_3, and the square transformation group is D_4. Find some common Bond between these groups and their makeup.

There is no common Bond between Lady Gaga's groups and her makeup.

14. **a.** Show that $(1\ 2)(1\ 3) = (1\ 2\ 3)$.
b. Show that $(1\ 2)(1\ 3)(1\ 4) = (1\ 2\ 3\ 4)$.
c. What's the next one?
d. Pick apart $(1\ 2\ 4\ 8\ 7\ 5)$ into five transpositions.

Two-element thingies like $(1\ 2)$ are called *transpositions*.

15. Rewrite $(1\ 3)(2\ 3)(2\ 5)(1\ 2)(2\ 5)(4\ 5)$ using a single set of parentheses.

The answer is *not* (132325122545).

16. A permutation is called *even* if it can be broken into an even number of two-element transpositions, and called *odd* if it can be broken into an odd number of two-element transpositions. Problem 15 contains an even permutation.

a. Of the eight permutations for the square transformation group, how many are even? How many are odd?
b. What happens if you combine two even permutations?
c. . . . two odd permutations? One odd, one even?

I've had a little bit too much, mods
All of these problems start to rush, start to rush by
How does he shuffle the cards? Can't stop and think, oh man
Where's my table? I broke my lamp, lamp
Just math. Gonna be okay.

17. These are the 24 permutations making up S_4, all the ways to go from one ordering of 1234 directly to another:

• $()$	• $(1\ 2)$	• $(1\ 2\ 3)$	• $(1\ 2\ 3\ 4)$
• $(3\ 4)$	• $(1\ 2)(3\ 4)$	• $(1\ 2\ 4\ 3)$	• $(1\ 2\ 4)$
• $(2\ 3)$	• $(1\ 3\ 2)$	• $(1\ 3)$	• $(1\ 3\ 4)$
• $(2\ 4\ 3)$	• $(1\ 4\ 3\ 2)$	• $(1\ 4\ 3)$	• $(1\ 4)$
• $(2\ 3\ 4)$	• $(1\ 3\ 4\ 2)$	• $(1\ 3)(2\ 4)$	• $(1\ 3\ 2\ 4)$
• $(2\ 4)$	• $(1\ 4\ 2)$	• $(1\ 4\ 2\ 3)$	• $(1\ 4)(2\ 3)$

This is not the same as in- and out-shuffling cards 1234. This is like having cards 1234 and just directly sorting them any way you want it, that's the way you need it.

a. Find all the even permutations. How many are there?
b. For each even permutation, write the result when 1234 is transformed by the permutation.

Your work in Problem 14 will help you decide.

18. You may or may not have made a list of all possible arrangements of the cards 12344321 with an unlimited supply of in- and out-shuffles.

One side of today's handout contains these arrangements. Draw arrows connecting the arrangements. Label each arrow with "I" or "O" to indicate whether the an in- or out-shuffle connects the two arrangements.

In this arrangement, the paired cards are considered equivalent, and the first half of the deck tells you what the second half must look like.

19. **a.** What two permutations are represented by the shuffles in Problem 18? Are they odd or even?

b. Using a d4, find a way to reproduce these two permutations visually. One permutation keeps "1" in place, and another keeps "3" in place.

Fun fact: Darryl likes Lady Gaga because she has monsters!

20. Explain why the group generated by in- and out-shuffles on 12344321 is isomorphic to A_4, the group of symmetries of the tetrahedron.

Alyssa says the A in A_4 stands for *Alyssa*, duh! Ok, actually the A in A_4 stands for *alternating*, meaning that every other element in S_4 is included. Since S_4 has 24 elements, A_4 has 12. A_n is made up of all even permutations of S_n.

21. Find four functions whose group under composition is isomorphic to the rectangle transformation group.

22. It sure is interesting that the number of orientations of a d6 is the same as the number of orientations of a d8. That suggests these two transformation groups might be isomorphic. What do you think?

It's got to be yes, because otherwise the question wouldn't be here, right? Boom! QED. More mathematicians should drop their chalk like in *Drumline* when they finish a proof.

Tough Stuff

23. It is always possible to write a permutation without re-using a element. When the $n!$ permutations of S_n are written this way, what fraction use all n elements? For S_4, 9 of the 24 transformations use all four elements.

Stop askin', stop askin', I don' wanna think anymore!

24. George Sicherman discovered a *different* way to populate 2d6 with positive integers, so that the sums of the two d6 matched the usual distribution. Neither d6 has the usual 1-6 on it. Only positive integers are allowed, and repetition is allowed.

2d6 means two six-sided dice.

a. Figure out what numbers are on the Sicherman dice.

b. Find all possible "Sicherman-like" dice for the d4, d8, d12, and d20. There may be more than one possible answer, or none at all! Woo hoo ha ha ha.

Wait, that's not a Gaga lyric!

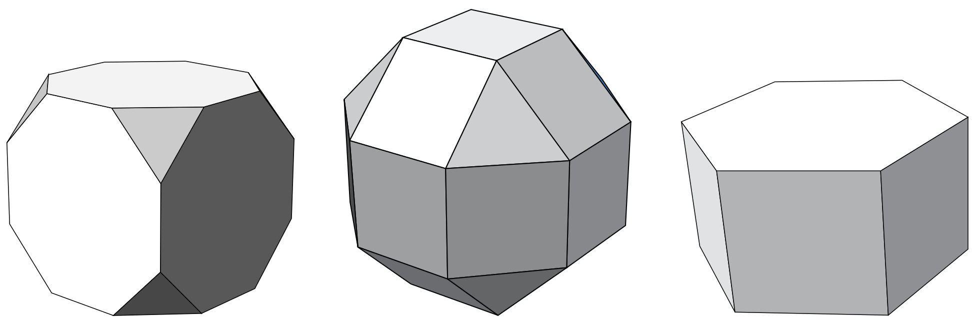
Problem Set 14: Some Math Camp That I Used To Know

Opener

Fill in this table.

Mwah ahaha! You thought you were done with tables?

Polyhedron	Faces (F)	Edges (E)	Vertices (V)	$F - E + V$
Cube	6	12		
Tetrahedron	4			
Truncated Cube				
Rhombicuboctahedron				
Hexagonal Prism				



Truncated Cube Rhombicuboctahedron Hexagonal Prism

Please be careful, and do not hit Ashli in the head with any more solids.

Important Stuff

1. For integers $a, b \geqslant 3$, find all possible solutions to the inequality
$$\frac{1}{a} + \frac{1}{b} > \frac{1}{2}$$

But you didn't have to come to Utah
Meet some friends and shuffle cards and then mod some numbers
I guess you've got to leave us though
Now we're just some math camp that you used to know

2. **a.** How many orientations does a cube have? Call this number n.
   b. Calculate each piece along with the total value of
$$\frac{n}{4} - \frac{n}{2} + \frac{n}{3}$$
   What do you notice?

3. A *Platonic solid* is made of regular polygons that meet the same way at all vertices. Suppose a Platonic solid has faces with a sides and vertices where b edges meet.

But you say it's just a friend.

a. If n is the number of orientations of this mystery solid, find the number of faces, edges, and vertices in terms of n.

Ooh, it's a mystery solid! Intriguing. Do tell more.

b. Rewrite this equation so that one side says $\frac{1}{a} + \frac{1}{b}$.

c. Find all possible solutions to the equation, along with the value of n for each. Oh snap.

d. How many Platonic solids are there, and what are the options?

Your Stuff

NM. **a.** Complete this equation in base 8.

$$0.\overline{7202013} + \boxed{} = 1$$

What will Darryl wear to Niagara Falls if he decides to become a stuntman? A *Darryl barrel*!

b. List all the powers of 2 in mod 1025 using only a four function calculator.

c. Find the base-5 decimal for 8/15.

2. Divide 20 by 7 and leave your answer in base 2.

Because it's an alien language, Jay knows!

10. Convert these base-10 numbers to base 5 now now now!

a. 73 **b.** $\dfrac{1}{25}$ **c.** $\dfrac{1}{2}$

NOW!

6. **a.** Prove $\frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$ using decimal expansions in base 10.

b. Prove $\frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$ using decimal expansions in base 3.

c. Prove $\frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$ using decimal expansions in base 2.

d. Compare the decimal expansion of $\frac{1}{3}$ in each of the three bases. What do you notice?

Only 3 of your base are belong to us! (Great joke, Kieran! You really had that one pegged.)

5. Andrea has an 8-card deck for some in-shuffling.

a. Track the location of each card. What cycles do you notice?

b. Armando has a 14-card deck. Repeat! What do you notice?

c. Go back to your 8-card deck and write out the order of all 8 cards after each in-shuffle. Explain the following observation.

$$\begin{aligned} 1 \cdot 5 &= 5 \bmod 9 \\ 5 \cdot 5 &= 7 \bmod 9 \\ 7 \cdot 5 &= 8 \bmod 9 \\ &\vdots \end{aligned}$$

Now and then I think of all the times you gave me Neat Stuff
But had me believing it was always something I could do
Yeah I wanna live that way
Reading the dumb jokes you'd play
But now you've got to let us go
And we're leaving from a math camp that you used to know

d. Does this pattern work for other deck sizes?

e. How can you use this pattern to determine the number of in-shuffles that it will take to restore an n-card deck?

12. How many in-shuffles does it take to restore a 12-card deck? What if you cut the deck into 3 piles? 4 piles? 6 piles? 5 piles?!

1 & 9. Suppose that while performing out-shuffles on a regular 52-card deck, you track whether a particular card appears on the left (L) or right (R) of the cut and you record that information.

a. If you notice that the card goes LLLRRLLLR, what card is it?

b. Repeat for LRLLRLLL.

c. The person writing the pattern for that last one might have messed up the last two letters. What should the last two letters be, and what card is it?

High five, you're a STaR!

d. Repeat for RLLRLR??.

e. Repeat for RRLLLR??.

3. Dowen and Barryl are avid Euchre players, so they want to try out their "card prediction" trick from last Monday using a 24-card Euchre deck (9 through Ace of each suit) instead of a 52-card deck.

MOAR BARREL

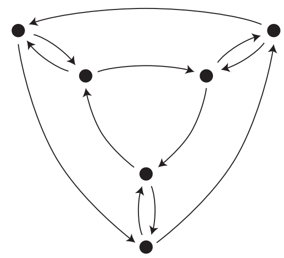
a. How many out-shuffles will it take to restore the deck to its original position?

b. What is 1/23 represented as a base-2 decimal?

c. What card positions does the 5th card take before it returns to its original position?

d. How many "Left-Rights" would you need to be told in order to predict the exact card that was chosen?

8. In what contexts has this graph come up during this class?



Rejected question: "Create an isomorphism between something we learned in class and any Georgia O'Keefe painting of your choice."

2. True or false: for all regular polygons, the number of symmetries/orientations is equal to twice the number of edges.

2. Make the most obnoxious table *evar*: an operation table for *all* the transformations of the cube, where the operation is "then."

Let's just be crystal clear we did not write this problem. Table 2 is asking you to do this. Not us.

11. **a.** Deborah loved Problem 3 from Set 12 so much that she redrew the graph using blue arrows for the in-shuffles and yellow arrows for the out-shuffles. Where have you seen a picture like hers before?

b. Gabe suggests making another picture. He says to draw the integers 0, 1, 2, 3, 4, an 5 and use yellow arrows to to connect two integers if you get from the first integer to the second by adding 3 (mod 6). Use blue arrows to connect two integers if you get from the first integer to the second by adding 2 (mod 6). Where have you seen a picture like his before?

Picture Gabe on stage singing some Beyonce. Better yet, don't.

c. Peter says something was missing from our parade construction. What was missing? Barb thinks this must be why we were getting confused looks from the crowd.

Put them together and what do you get? Bibbity boppity group. Symmetric across a Z La sub3 group la group. (Uh, *what*?)

12. Oh *noes*! A feared red math germ has begun infecting PCMIers! Kieran suggests that it might be possible to come up with a cure by shuffling the red math germ's genes to create an antidote. He lists the genes as "r e d m a t h g e r m" and then transforms its genes using $(1\ 2)(3\ 11)(10\ 9\ 8\ 7\ 6)$. What do you think of Kieran's suggestion?

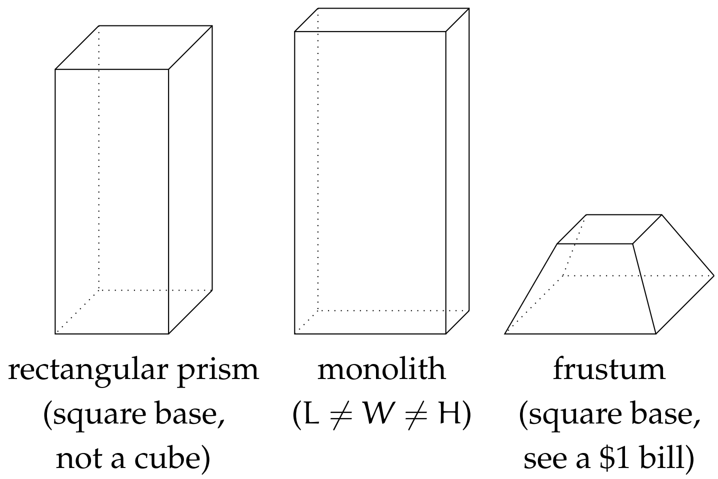
Really, Table 12, really? *Really?* I guess it's true, we can't stop you.

Hey! This is math camp, not virology camp!

10. Refer back to Problem 1 from Set 13. Find two counterexamples to the statement "You can choose any two transformations and always achieve all eight symmetries/orientations."

12. How many different transformations can you perform on these non-platonic solids, while still making sure that they occupy the same volume of space?

If they're non-platonic, does that mean these solids hooked up? Zig a zeg ahh.



Don't let these problems frustumrate you!

7. **a.** If you number a 52-card deck from 0 to 51 and track the position of card #1 during out-shuffles, you will get this cycle: $S = \{1, 2, 4, 8, 16, 32, 13, 26\}$. Make an operation table for these numbers, where the operation is multiplication mod 51.

b. Show that this group is isomorphic to $(\mathbb{Z}_8, +)$.

c. Choose a different cycle of cards by tracking a different card as the deck is out-shuffled. Does that set of numbers also form a group under multiplication mod 51?

This problem is as awesome as Fred's name tag. It's like the Bohemian Rhapsody of math. Everybody knows that song!

11. **a.** Put the numbers $1, \ldots, 12$ on the edges of a cube so that the four numbers around each face have the same sum.

b. Find 19 other really different ways to do this, where "really different" means not a cube symmetry of another solution. Hint: Everybody's shuffling.

Our Stuff

5. Show that S_4 is isomorphic to the group of symmetries of the cube by finding four interesting axes of symmetry.

4. Show that a transposition cannot be both even and odd at the same time.

3. **a.** How many *distinct* ways are there to number the four faces of a d4? By this we mean that you can't find a symmetry that brings one to another. Wow, that's not many!

b. How about for a d6?

c. How about for a d8? Wow, that's . . . not not many?

2. The number of orientations of a d12 and a d20 is the same. Are these isomorphic doyathink?

1. Figure out how to use shuffling to find the *base-10* decimal expansion of $\frac{1}{51}$ along with other fractions.

No More Stuff

Balderdash time! Six of the seven definitions below are false. Which is the right one?

Richard says that the S in S_4 stands for *Schrödinger*, a man who owned too many cats.

Tina says that the S in S_4 stands for *scaler*, someone who climbs a large mountain in Park City.

Mark says that the S in S_4 stands for *subgroup*, a set of nuclear wessels.

Rina says that the S in S_4 stands for *set*, something you win at tennis by finding three cards with different properties.

Rebecca says that the S in S_4 stands for *simplex*, the ability to make things easier and harder at the same time.

Robert says that the S in S_4 stands for *seecant*, someone who really needs glasses.

Jodie says that the D in S_4 stands for *symmetric*, meaning . . . you know, symmetric!

Don't you forget about us
We'll be alone, shufflin', you know it baby
These groups, we'll take them apart
Then put 'em back together in parts, baby

I say $(\mathrm{LA})^{55}$

When you walk on by

Will you call me maybe . . .

(See you again soon.)