Paper Cup Mathematics—Directions for the Teacher

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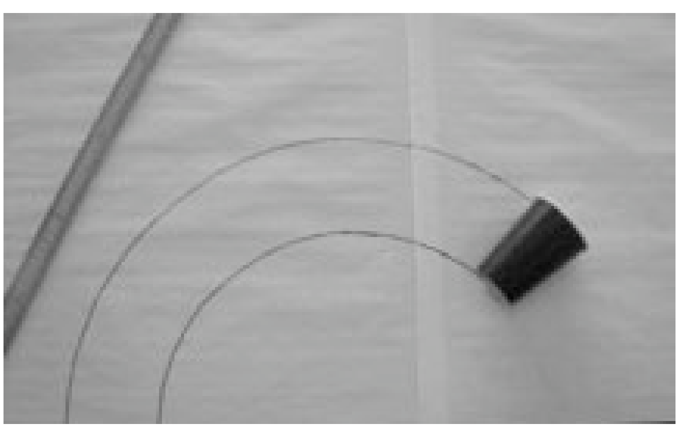
CONCEPTS: Geometry, Measurement
SKILLS: Making constructions, finding the area of an annulus (the "ring" formed by two concentric circles of different radii), making measurements, problem solving
GRADE LEVELS: 8–12
MATERIALS: For Worksheet 1: Paper cups, compass, straightedge, chart paper, scissors, rulers, Student Worksheet 1
For Worksheet 2: *Geometer's Sketchpad*®, Student Worksheet 2

DESCRIPTION
Imagine a discarded paper cup rolling on its side on a flat surface. What geometric shape does the path of the cup form?

Students investigate this question and begin the activity by experimenting with paper cups. They devise methods to draw the path of the cup, which may include a compass and straightedge construction. The activity concludes with a problem that requires finding the area of a carousel platform. Student Worksheet 2 includes an investigation using *Geometer's Sketchpad*®.

DIRECTIONS
Demonstrate the situation of rolling a paper cup on its side on a flat surface to the class. Ask students to formulate a conjecture about the path of the cup. Distribute Worksheet 1, paper cups, and chart paper to groups of students and set the task to determine the geometric shape formed by the rolling cup as pictured in Figure 1.

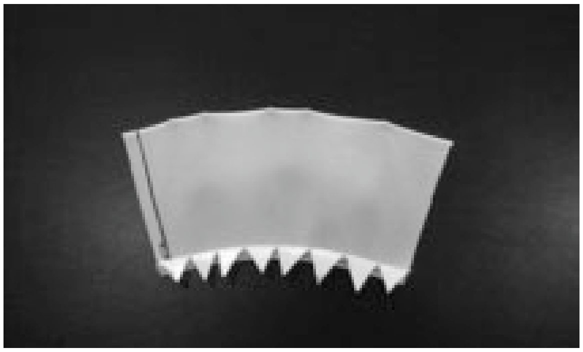
Figure 1: A rolling paper cup and its path



Some groups may need help getting started. If so, suggest that they start rolling the paper cup on one edge of their chart paper and try to determine the path.

When the cup is taken apart at (or cut along) its seam and flattened out, the cup may look like the cup in the photo of Figure 2.

Figure 2. Photo of the "side" of a flattened paper cup with the bottom removed



Students can trace the shape over and over until the path that would be formed by the rolling cup is closed. Once students are convinced that the path is determined by two circles (the top and bottom rims of the cup) and the distance between them (the slant height), they should try a "proof by construction." To do this, they may use one flattened cup and, with their construction tools, draw two chords on an arc of the outer (or greater) circle and construct the perpendicular bisectors of the chords. Where the perpendicular bisectors of the chords intersect is the expected center of the concentric circles forming the original path. The diameters of the outer and inner circles may be measured and the area of the region between the circles can be estimated.

After students have finished Worksheet 1, there are many other investigations that can be posed.

- Estimate the diameter of the outer circle by using only the measurements of the cup. What process do you follow to do this?
- Describe how to design a cup that will roll out a circle of a given radius if it can be done.
- Describe how to design a cup that will roll out a specific area if it can be done.
- Identify, if possible, paths determined by rolling the following flattened shapes (ignoring their bases) on a flat surface: a square pyramid; a hexagonal pyramid; a cone.
- Suppose you tape a sector of an annulus together to form a cup. Explain how to find the surface area of the outside of the cup formed including its bottom.

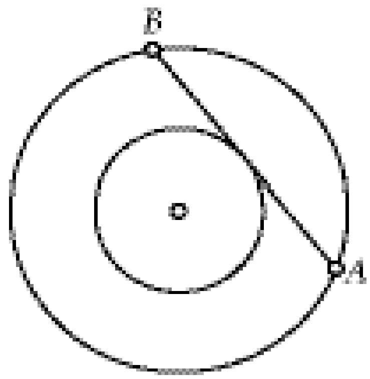
EXTENSION: A PUZZLE ABOUT ANNULI

The following problem is adapted from a puzzler posed on Car Talk, a NPR radio show. For a complete discussion of the original problem, log onto http://www.cartalk.com/content/puzzler/transcripts/200218/index.html.

The same problem was posed as the area of a lighthouse keeper's rug in "Creative Geometry Problems Can Lead to Creative Problem Solvers," by George A. Milauskas in *Learning and Teaching Geometry, K-12*, the 1987 Yearbook of the National Council of Teachers of Mathematics.

Pose the following problem to your students. Their group was hired to paint the floor of a merry-go-round. They want to measure the area of the floor exactly because they do not want to buy extra paint. The carousel is circular and in the middle is a smaller circle that contains all the machinery for the carousel. Therefore, the carousel platform is an annulus. The only measurement provided is the length of the chord of the outer circle that is also tangent to the inner circle. (See Figure 3.) The measure of segment *AB* is 70 feet. Find the area of the annulus.

Figure 3. Concentric circles with a chord of the greater tangent to the smaller



Ask students to simulate the Car Talk annulus problem using dynamic software such as *Geometer's Sketchpad®*. Have them begin with a chord that remains constant when the center of the circles is dragged. Worksheet 2 details the steps of the construction.

References

http://www.cartalk.com/content/puzzler/transcripts/200218/index.html.

Milauskus, George A. "Creative Geometry Problems Can Lead to Creative Problem Solvers," in *Learning and Teaching Geometry, K-12*, 1987 Yearbook of the National Council of Teachers of Mathematics (edited by Mary Montgomery Lindquist and Albert P. Shulte). Reston, VA: The Council, 1987, p. 77.

Answers to Worksheet 1

2. The area is dependent on the cup used. The area of the whole annulus is the difference of the areas of the two circles. The radii of the circular paths can be computed from the dimensions of the cup (though you may not choose to have your students do this). If R is the radius of the outer circle of the annulus and r is the radius of the inner circle, then $R = r + s$, where s is the slant height of the cup. Then if C and c are the circumferences of the top ring and bottom ring of the cup, the ratio of the radii and the ratio of the circumferences are equal, so $(r + s)/r = C/c$. This can be solved for r and then for $R = r + s$.
3. The number of times the flattened cup has to be traced to complete the annulus is 360°/(measure of the arc of the outer circle in degrees).
4. The number will not change. The same process as in Number 3 determines the number of times the flattened cup must be traced.

Answer to the Problem of Car Talk and Worksheet 2

The area of the annulus is independent of the radii of the concentric circles. In this particular with a chord of length 70 ft, the area is 1225π ft^2. This can be seen by using the Pythagorean theorem with the two radii and 35 ft (half of the chord) and the fact that the area of the annulus is the difference of the areas of the two circles and so is proportional to the difference of the squares of the radii.