Session 13, July 25
Applications of Gaussian Integers

1. Use the division algorithm in \( \mathbb{Z}[i] \) to find a valid quotient and remainder when \((5+7i)\) is divided by \((3+i)\).

2. Use the Euclidean algorithm in \( \mathbb{Z}[i] \) to find a GCD of \((5+7i)\) and \((3+i)\). How is it possible for there to be more than one GCD?

A lattice structure can be used to picture the division algorithm in \( \mathbb{Z}[i] \) (see handout).

3. Use a lattice on graph paper to find the quotient and smallest possible remainder when \((8+12i)\) is divided by \((3+2i)\).

Recall that the norm of a Gaussian integer \( z = (a+bi) \) is \( N(z) = a^2 + b^2 \), and the norm is the square of the magnitude (or modulus) of that Gaussian integer.

4. Explain why the norm must be an integer. Can the norm ever be a perfect square?

5. Show that if two Gaussian integers are multiplied, their norms are multiplied. In other words, given two complex numbers \( x \) and \( y \), show that \( N(xy) = N(x)N(y) \).

One consequence of Problem 5 is that the square of any Gaussian integer must have a square number as its norm.

6. For each of these Gaussian integers, plot the Gaussian integer, then find and plot its square: \((1+i), (2+i), (3+2i), (1+4i), 3, 2i\). Graphically, what relationships are there between a Gaussian integer and its square?

7. For each of the square Gaussian integers in Problem 6, construct a right triangle (if possible) with the square as the hypotenuse. What can be said about this triangle? What happens in those cases where a right triangle cannot be constructed?

A second consequence of Problem 5 is that the norm of a square Gaussian integer allows the equation \( a^2 + b^2 = c^2 \) to be solved with integers for \( a \), \( b \), and \( c \). The values for \( a \) and \( b \) are the real and imaginary parts of the Gaussian square, and the value for \( c \) is the magnitude.

So, one way of finding Pythagorean triples is to start with any Gaussian integer, square it, and yabba dabba doo, there’s a triple.

1. Generate a Pythagorean triple from the Gaussian integer \( 3+i \). Generate a triple from \( 7+4i \). Generate a triple from \( 22+19i \). Generate a triple from \( 8+11i \).

2. Generate a Pythagorean triple from the Gaussian integer \((m+ni)\). Note that this will give you a general formula for finding an unlimited plethora of Pythagorean triples.

One issue which you may have noticed is that the Pythagorean triples are not always “primitive”; that is, the numbers \( a \), \( b \), and \( c \) are not relatively prime. Additionally, multiplication of Gaussian integers makes it possible for \( a \), \( b \), or both to become negative; not a good thing for the sides of a triangle.
10. What must be true of \(m\) and \(n\) if \((m + ni)\) generates a primitive Pythagorean triple? Alternatively, what must be true of \(m\) and \(n\) if \((m + ni)\) does not generate a primitive triple?

11. If \(m\) and \(n\) are both positive, what must be true of \(m\) and \(n\) for all the numbers in the generated triple to be positive? Your work in either Problem 6 or 9 can be used here.

12. Use multiplication of Gaussian integers to find all Pythagorean triples (not just primitives!) with hypotenuse \(c = 25\); with \(c = 35\); with \(c = 65\); with \(c = 125\). Use the property from Problem 5 here!

13. Which prime numbers can be the hypotenuse of a Pythagorean triple? Are all these triples primitive?

14. Which prime numbers can be the norm of a Gaussian integer?

15. What are the prime numbers in \(\mathbb{Z}[i]\)? Plot them in the plane, up to norm 50.

One of the reasons the process for finding Pythagorean triples works is because of factoring in \(\mathbb{Z}[i]\):

\[
\begin{align*}
c^2 &= a^2 + b^2 \\
c^2 &= (a + bi)(a - bi)
\end{align*}
\]

If such a factoring is possible, then the norm of \((a + bi)\) must equal \(c^2\); and \(c\) is the length of the third side of the triangle.

A similar factoring allows for the generation of Eisenstein triples, which are numbers which form the sides of a triangle with a 60-degree angle. The Law of Cosines gives us the relationship between the three sides:

\[
\begin{align*}
c^2 &= a^2 + b^2 - 2ab \cos 60 \quad (1) \\
c^2 &= a^2 + b^2 - ab = N(c) \quad (2)
\end{align*}
\]

(Note: the right side of (2) is the quadratic factor of \(a^3 + b^3 = (a + b)(a^2 - ab + b^2)\).)

Factoring gives \(c^2 = (a + b\alpha)(a + b\overline{\alpha})\), with the new number \(\alpha\) having the property that \(\alpha^2 + \alpha + 1 = 0\). (This is similar to \(i\), where \(i^2 + 1 = 0\).) Note that \(\overline{\alpha}\) is the conjugate of \(\alpha\). It is more typically written that \(\alpha^2 = -\alpha - 1\) since this form is more useful in algebraic simplification.

It takes a little getting used to this new number, although \(\alpha\) is actually a member of \(\mathbb{C}\). (Specifically, it is the number \(\text{cis} 60\)). \(\alpha\) is used in the best-known proof that there are no solutions to the equation \(x^3 + y^3 = z^3\). Numbers in the form \(z = a + b\alpha\), where \(a\) and \(b\) are integers, are written as \(\mathbb{Z}[\alpha]\).

16. Verify, using the property of \(\alpha\), that the factoring \(c^2 = (a + b\alpha)(a + b\overline{\alpha})\) is valid.

Now, generating Eisenstein triples is done the exact same way we generated Pythagorean triples. Choose any element in \(\mathbb{Z}[\alpha]\) and square it. Don’t forget that the property of \(\alpha\) is not the same as the property of \(i\).
17. For each of these, find the square and the corresponding Eisenstein triple:
\[(1 + \alpha), (2 + \alpha), (3 + 2\alpha), (1 + 4\alpha), 3, 2\alpha.\]

18. Generate an Eisenstein triple from the number \((m + n\alpha)\). This will give a general formula to generate Eisenstein triples.

As with Pythagorean triples, generating primitive triples becomes an issue, as does generating triples which have positive legs.

19. Describe conditions on \(m\) and \(n\) which will lead to primitive Eisenstein triples with positive side lengths.

20. Which prime numbers can be the “hypotenuse” (side opposite the 60-degree angle) of an Eisenstein triple? Of a \textit{primitive} Eisenstein triple?

21. Beginning with the Law of Cosines, find a formula that will generate triangles with one angle of 120 degrees.