1 *Day 1: The Basics (?)*

Hey, welcome to **"The Art and Craft of Adding and Subtracting"**. The intent of these sessions is ostensibly to learn more math, but we really think it's about a teaching philosophy. We don't expect you to get all these problems right; if you can get every problem on every set, we haven't done our job. We are confident that you will learn more math by doing these problems than by us standing at the board at 8:20 am every day for two hours. Hopefully you will agree, and it may help you in your teaching (or, better yet, in others' teaching!). Most of all, have a good time and don't worry about being stuck on a problem. Thanks and enjoy!

It must be important if it's in bold.

Keep on Adding A bored student in a math class started fiddling with the calculator. Alice started with the number 3, then kept on typing +5, +5, +5. Little did she know she was performing *iteration*, or repeated operation, or that her work starts a journey:

1. Make a table, starting with 0, for the number of times Alice performed the +5 operation and the end result of the calculation.

Iterations	Result
0	3
1	
2	
3	
4	
5	
6	

2. Suppose Alice's result when the teacher finally noticed was 188. How many times did she perform the operation? How did you solve this problem?

A "recursive" formula for Alice's operation is

$$f(0) = 3 \qquad f(n+1) = f(n) + 5$$

An alternate version that is equally good is:

$f(n+1) - f(n) = 5$.

This describes what she did to get from one step to the next. The recursive formula isn't that useful, though, if we wanted to find out the value of $f(210)$.

3. Use your work in problem 2 to write a "closed-form" version of Alice's formula. That is, make a formula that says $f(n) = \cdots$ directly.

4. If Alice started with the number a and added b each time, what would the recursive formula be? What would the closed-form formula be?

Later, Alice wrote down the numbers

$$1 \qquad 2 \qquad 3 \qquad 4 \qquad 5$$

then decided to add them in pairs and write the sum underneath. So under those numbers she wrote

$$3 \qquad 5 \qquad 7 \qquad 9$$

... which is the sum of each pair of successive numbers.

5. What's the last number generated this way? Make sure you keep track of this table, you might see it again.

6. In terms of the numbers written in the first row, how is the last number generated? Is there a formula?

Take a few minutes on this.

7. Alice started over, this time with the numbers

$$1 \qquad 10 \qquad 100 \qquad 1000 \qquad 10000$$

What's the last number generated this way? Does it give any insight on a possible formula?

8. Find the last number if Alice started with

$$1 \qquad 2 \qquad 3 \qquad 4 \qquad 5 \qquad 6$$

Can you get it using a formula from the original numbers?
Can you get it using the information from problem 5?

Plenty of food for thought here, and all we did was add!

Keep on Subtracting Bob was presented with a table of information and asked to find the formula. He scoffed at the word "the" and decided to do some other stuff. Here's the table of information:

Input	Output
0	3
1	8
2	13
3	18
4	23
5	28
6	33

9. Find a formula that fits this table. Is this the only formula that fits the table? If yes, explain why; if no, explain why not, and see if you can find another.

Bob (bored or not) decided to take successive outputs and subtract them from one another; this is sometimes referred to as taking the *common difference*.

10. Calculate the common differences for this table.

As a matter of notation, the numbers you generated are the common difference at *zero* through *five*.

11. Write the common difference at zero in terms of the function's inputs. In other words, fill in the blanks on this:

$$d(0) = f(\quad) \qquad f(\quad)$$

12. Is there a formula for the common difference at n in terms of the function's inputs? Is there a relationship to Alice's recursive formula?

And no, "5" is not the answer here.

13. Could Bob have calculated the *second common differences* (differences of the differences)? How far could he go with this?

Bob wondered if this worked the other way, too; if he had the list of common differences, could he figure out what the original numbers were?

14. Suppose Bob knew the common differences are 2, 5, 12, and 28. Is this enough information to tell you what the five outputs were? Suppose Bob also knew the first output was 1. Would that help?

15. Suppose the common differences are 62, 13, 27, and 38 and Bob knows the first output is 10. Calculate the final output as fast as you can.

16. Suppose the common differences are 27, 62, 38, and 13 and Bob knows the first output is 10. Calculate the final output as fast as you can. What's going on here? Can you explain it? Prove it?

Later, Bob received another table:

Input	Output
0	1
1	3
2	8
3	20
4	48

This was a little more interesting, since he could "keep going" with the differences until there was only a single number left.

17. What's the last number generated this way? Make sure you keep track of this table, you might have seen it again.

18. In terms of the numbers written in the second column (i.e., the original outputs), how is the last number generated? Is there a formula?

Take a few minutes on this.

19. Bob started over, this time with the numbers

1 10 100 1000 10000

What's the last number generated this way? Does it give any insight on a possible formula?

Keep on... Both! When Alice and Bob compared notes at lunch, they were shocked and chagrined to find their boredom had led to mathematical insight, and that their work had been so close together. A teacher recommended they organize things this way:

Input	Output	Δ	Δ^2	Δ^3	Δ^4
0	1	2	3	4	5
1	3	5	7	9	
2	8	12	16		
3	20	28			
4	48				

Depending on which way you look, this table is either adding *or* subtracting. If you only know the outputs, you can subtract and determine the differences. The *iteration* is still going on, but this time the thing being iterated is the operation of taking common differences. So, the exponent 2 in Δ^2 doesn't mean "squaring," but repeating the Δ operation twice.

On the other hand, if you only know the first row (the sequence of common differences) you can add your way down and over to find more information. Alice and Bob have both had opportunities to add, and this table includes all that information.

20. On this table, circle the number 16, then shade the two numbers that Alice added to get 16 in problem 5.

21. On this table, circle the number 48, then shade the five numbers, including the "1", that Bob added to get 48 in problem 7.

These will be referred to as the "up and over" and "hockey stick" properties of these tables:

- **Up and over**: Any number in the table (other than inputs) can be calculated as the sum of two numbers – the number directly above it and the number directly to the right of the one above it.

- **Hockey stick**: Any number in the table (other than inputs) can be calculated as the sum of the top value in its column and *all* the numbers above it in the column to its right.

22. Find a function that agrees with this table:

Input	Output	Δ
0	c	d
1		d
2		d
3		d
4		d
5		

23. Find a function that agrees with the table below.

You might want to fill in the table first. Is there only one table that agrees with these "'boundary conditions?"

Input	Output	Δ	Δ^2
0	3	4	2
1			2
2			2
3			2
4			
5			

24. Find a function that agrees with this table:

This one is pretty tough, for now.

Input	Output	Δ	Δ^2
0	a	b	c
1			c
2			c
3			c
4			
5			

25. Suppose you knew that in the table you used in problems 20 and 21 on page 5, the Δ^4 column was *always* "5". Use that information to find $f(10)$ as quickly as possible. Might there be a formula for this?

Boy, wouldn't *that* be nice.

2 *Morning Addition*

In order to keep Alice and Bob a little more busy, the teacher tells them to join Karl in calculating the sum of the first n whole numbers. To make it "easier," she tells them to refer to this function as $f(n)$.

1. What is $f(1)$, the sum of the first 1 whole numbers?

Hint: The answer is *not* 1. (Whole!)

2. Find $f(n)$ for integer n between 0 and 10. Think about what you did while you were filling out the table. Yes, $f(0)$ is zero, since you didn't add anything and zero is the additive identity.

3. Alice figured out a "recursive" formula for this function:

$$f(n+1) = f(n) + \cdots$$

Figure out the missing piece of the formula.

4. Draw a plot of $f(n)$ for n between 0 and 6. What kind of graph is it? What does that imply about the *degree* of the function involved?

Looking up from their work, Bob and Alice saw their teacher introduce the *Factor Theorem*.

Theorem 1 (Factor Theorem) *If $f(x)$ is a polynomial function, and $f(a) = 0$, then $(x-a)$ is a factor of $f(x)$. Put another way, the function can be written as*

$$f(x) = A(x-r_1)(x-r_2)\cdots(x-r_m)$$

where r_1 through r_m are roots *where $f(r_j) = 0$.*

For example, if I know $f(3) = 0$ and $f(5) = 0$, then $(x-3)$ and $(x-5)$ must be factors of the polynomial. There might be others, of course. If I know these are the *only* factors, then the polynomial must be $A(x-3)(x-5)$, and then I just have to find A.

5. For the function Bob and Alice were working on, what are the roots? How do you know? What are the factors, then? Explain how problem 4 suggests that these are the only two factors.

6. Use the information about the roots to find a "closed-form" formula for $f(n)$, then use it to find $f(101)$.

Karl looks on, impressed. He asked about what Alice and Bob were working on yesterday. Bob shows him some problems that not everyone might have gotten to.

7. Suppose Bob knew the first differences are 8, 23, 64, and -7. Is this enough information to tell you what the five outputs were? Suppose Bob also knew the first output was 1. Would that help?

8. Suppose the first differences are 62, 13, -27, and 38 and Bob knows the first output is 10. Calculate the final output as fast as you can.

9. Suppose the first differences are -27, 62, 38, and 13 and Bob knows the first output is 10. Calculate the final output as fast as you can. What's going on here?

10. Build a table of differences, second differences, and third differences for this function $f(n)$. What do you notice about the first differences? The second differences? The third differences? Bob provided Karl with a template.

Input	Output	Δ	Δ^2	Δ^3
0	0			
1	0			
2	1			
3	3			
4	6			
5	10			
6	15			

By the way, the letter Δ stands for "difference" here, and the exponent 2 in Δ^2 doesn't mean "squaring," but repeating the Δ operation twice.

Alice became interested in the rows, rather than the columns

– especially the row with $f(0)$ and its differences. This row was all zeros, except for a 1 as the second difference. More food for thought? A good thing, since lunch was a long way off.

11. Construct a new table where the row with $f(0)$ is all zeros, except for a 5 as the first output. What function is this?

12. Construct a new table where the row with $f(0)$ is all zeros, except for a 1 as the first difference. What function is this?

13. Construct a new table where the row with $f(0)$ is all zeros, except for a 3 as the first difference. What function is this? How is this related to your answer to problem 12?

14. Construct a table where the row with $f(0)$ is all zeros, except the first output is a and the first difference is b. What function is this? How is this related to your answers to problems 11 and 12?

15. Construct a new table where the row with $f(0)$ is all zeros, except for a 2 as the second difference. Use your answer from problem 6 to answer as quickly as possible: What function is this?

16. Imagine a new table where the row with $f(0)$ is all zeros, except for a 1 as the first difference *and* a 2 as the second difference. **Don't construct it!** Instead, predict the function based on your answers to problems 12 and 15 *before* you construct it. Okay, now construct the table. Were you right?

By *all* zeros, we literally mean forever. This means you can keep the table going as long as you want without surprises. Oh, and even though there is more than one answer to the "Which function?" questions, you'll give the right one.

Alice was pretty impressed. She figured she could now predict any function that went all the way up to second differences. This was going to make some of her homework *much* easier.

Remember, Δ refers to the differences, and Δ^2 refers to the second differences.

17. A function has $f(0) = 3$, $\Delta(f)(0) = 4$, $\Delta^2(f)(0) = 2$, and all later differences are zero for all numbers. What is the polynomial function that matches this table? Test your answer by checking your prediction for $f(5)$ against the value you got from the table in yesterday's problem 23. (Didn't do it? Don't worry.) Easier now, ain't it?

So, $\Delta(f)(0) = 4$ means the first difference at zero is 4. Get it?

18. A function has $f(0) = a$, $\Delta(f)(0) = b$, $\Delta^2(f)(0) = c$, and all later differences are zero. What is the function? Don't worry about expanding or collecting terms.

At this, Kurt came by and mentioned that the same idea might be used to figure out the polynomial to use for third differences. This would add to Alice's list of polynomials for everything up to second differences.

19. What happens to the degree of a polynomial each time you perform the "difference" operation? Give some examples. Can you convince yourself it's always true? Can you convince *someone else*?

i.e., prove it?

20. Suppose you knew the second differences were all the same number. What kind of polynomial did you start with?

21. Suppose you knew the third differences were all the same number. What kind of polynomial did you start with? How many *roots* does this kind of polynomial have?

22. Construct a new table where the row with $f(0)$ is all zeros, except for a 1 as the *third* difference. If this is a polynomial fuction, what is its degree? Calculate all the rows out to $f(8)$ and its differences.

23. What are the *roots* of the polynomial in problem 22? How do you know? What are the factors? Use this information to find the function that fits this table. Don't worry about expanding or collecting terms, just leave it factored.

A *root* is a place where the function is zero.

24. A function has $f(0) = a$, $\Delta(f)(0) = b$, $\Delta^2(f)(0) = c$, $\Delta^3(f)(0) = d$, and all later differences are zero. What is the function? Don't worry about expanding or collecting terms, since we are looking for a pattern here.

25. Alice made up a table with $f(0) = 3$, $f(1) = 10$, $f(2) = 23$, and $f(3) = 48$. She was able to find a cubic polynomial to fit this table *very* quickly. Can you?

26. As a joke, Bob added on $f(4) = -5$. Can you find a polynomial to fit this table? How quickly? You'll have to extend the reasoning used in the last few problems, but if you've gotten here, you must be good at that.

Alice and Kurt felt pretty happy about this, and even Karl agreed that this would be a powerful tool. Bob had other ideas, looking at some of the other rows in the table from problem 22. Bob could swear he'd seen some of these numbers before.

Still going? Okay, here's some more.

27. Suppose $f(0) = 1$ and all differences at zero are 1. Yes, forever. What function is this?

28. Suppose $f(n) = 10^n$ for all n. What are its first differences? second differences? Use the definition of the Δ function to explain why this is true.

29. Suppose $f(0) = 1$ and each difference is ten times larger than the one before it. What function is this? Can you prove why? What if each difference is n times larger than the one before it?

cough... *binomial*... cough...

30. Use what you've learned to find the sum of the first 100 fourth powers without calculating them all.

31. Suppose $f(0) = 1$ and each difference is the opposite of the one before it (so it's $1, -1, 1, -1, ...$). How many roots does this function have?

3 *Blazing Through*

It's time to review! Hooray! Take some time and make sure you're able to do this kinda stuff.

1. Find a polynomial function with $f(0) = 5$, $f(1) = -3$, and $f(2) = -11$. Use differences to help!

2. Find a polynomial function with $f(0) = 21$, $\Delta f(0) = -18$, and $\Delta^2 f(0) = 6$.

Alice and Bob hear (but don't particularly listen to) the teacher talking about *combinations* and *permutations*. A combination is a group in which the order does not matter, while in a permutation, order matters. The "Pick-6" style of lottery is a combination. Choosing what shirts to wear over the next five days is usually a permutation problem, unless you really, really like the same shirt.

3. How many ways are there to pick a group of three people out of eight? By "group," it means "you three" and not "you first, you second, then you third." How many ways are there to pick a group of two out of eight? A group of one out of eight? A group of zero out of eight? How about a group of three people out of two?

4. Elbonia has a "Pick-1" lottery, where the player picks a single number and hopes it matches the drawing. How many different possible plays are there in this lottery if there are 49 numbers in the drawing?

5. Freedonia has a "Pick-2" lottery, where the player picks two numbers and hopes they match the drawing (in either order). How many different possible plays are there in this lottery if there are 49 numbers in the drawing? if there are n numbers in the drawing?

The distinction *either order* makes this a combination problem, rather than a permutation problem.

6. Coruscant has a "Pick-3" lottery. Same deal. How many different plays in this lottery for 49 numbers? for n numbers?

The notation $\binom{n}{k}$ is used to represent combinations. Alice and Bob's teacher starts talking about drawn-out formulas for combinations, and as usual, they tune out and resume the stuff they were working on yesterday.

So, the number of plays in a "Pick-6" lottery with 49 numbers is $\binom{49}{6} = 13,983,816$.

7. If you didn't do problem 22 from yesterday, do it now:

Construct a new table where the row with $f(0)$ is all zeros, except for a constant 1 as the third difference for all entries. If this is a polynomial function, what is its degree? Calculate all the rows out to $f(8)$ and its differences.

8. Look at the rows of this table, starting with $f(3)$. Do the four numbers in this row look familiar at all? What about the four numbers in the next row? How about the numbers in the $f(8)$ row?

9. Consider the four numbers in the row starting with $f(3)$. Explain how these numbers are used to generate the row starting with $f(4)$. Does this process seem familar at all?

10. What about the numbers for $f(n)$, from $n = 0$ onward? Do these numbers look familar? How so?

They might not, of course!

Hey, it's fine if you just said, "No, no, NO!!!" to problems 8, 9, and 10. Bob found some patterns, though, so hopefully you did too. Now, for the flip side.

11. **Do this problem by completing the table, even if you know how to get the formula for $f(x)$.** A function has $f(0) = 3$, $\Delta(f)(0) = -4$, $\Delta^2(f)(0) = 5$, and constant $\Delta^3(f)(0) = 2$. Complete the table to find $f(8)$.

The bold thing would've been a side note, but Art would ignore it.

12. Now, explain how $f(8)$ could be calculated from the two numbers directly above it (remember, "up and over"). Write $f(8)$ in terms of these numbers.

13. Now, explain how $f(8)$ could be calculated from the three numbers directly above that. Write $f(8)$ in terms of these numbers, and collect terms if you used the same one more than once. You'll get the most benefit by writing

$$f(8) = m \cdot \_\_\_\_ + n \cdot \_\_\_\_ + \cdots$$

even if some of those m and n are ones.

14. Now, explain how $f(8)$ could be calculated from the *four* numbers directly above that. Write $f(8)$ in terms of these numbers, and collect terms if you used the same ones more than once. Follow the advice in problem 13; it will help to see the overall pattern here.

15. Now, explain how $f(8)$ could be calculated from the *five* numbers directly above *that*. Write $f(8)$ in terms of these numbers, and collect terms if you use the same one more than once (and you will).

What? You only see four? Too bad, there are five – to get the fifth one, you're going to need a bigger table, to paraphrase Roy Scheider.

16. Finally, explain how $f(8)$ could be calculated from *nine* numbers calculated across the top (starting with $f(0)$).

To get the fifth and higher terms, you'll need a bigger table or a pattern from the previous problems.

Alice showed her work to her classmate Pasquale, who really seemed to appreciate these numbers. Pasquale told Alice that if she'd paid attention in class, she might've learned a more convenient way to write the numbers in problem 16 – something to do with a triangle. Whatever.

17. Write $f(8)$ in terms of the combination numbers $\binom{8}{k}$ and the values across the top row: $f(0)$, $\Delta(f)(0)$, and so forth. What does this have to do with Pasquale's triangle?

18. Find the value of $f(10)$ as quickly as possible.

19. How would you find $f(97)$? Find it if you want to, but use a calculator!

20. Find the value of $f(5.5)$ as quickly as possible.

Problem 20 presented a problem (literally) to Alice, since there weren't gaps in the table and there wasn't any way to say how many ways to to pick two people out of a group of 5 and a half. Bob and Kurt felt fine with it, since they have polynomials that they could add to find the answer:

$$f(x) = 3 \cdot 1 + (-4) \cdot x + 5 \cdot \frac{x(x-1)}{2} + 2 \cdot \frac{x(x-1)(x-2)}{6}$$

These polynomials come from yesterday's work. The first difference at zero is -4, the second difference is 5, and so on.

These polynomials (1, x, $\frac{x(x-1)}{2}$, etc.) are called the *Mahler polynomials*. $\frac{x(x-1)}{2}$ is referred to as the second Mahler polynomial (since it is of degree 2). The Nth Mahler polynomial has roots of 0, 1, 2, 3, …, $n-1$, and $f(n) = 1$. This helps these polynomials form a *basis*. They can be added to one another very easily to form a polynomial matching a difference table.

21. What is the next (fourth) Mahler polynomial? What about the fifth? What's a formula for the Nth Mahler polynomial?

22. Use Mahler polynomials to find a polynomial with this table. You'll have to take differences first.

Input	Output
0	6
1	5
2	24
3	99
4	290
5	681

23. What is the sum of the first three square numbers? Oh, wait, it's 14. Never mind. What is the sum of the first n square numbers? If $f(n)$ is the sum of the first n square numbers, build a table for $f(n)$ from zero to at least eight.

As before, the sum of the first 0 square numbers is 0, since you haven't added anything.

24. Use Mahler polynomials to find a formula for the sum of the first n square numbers.

25. Use Mahler polynomials to find the sum of the first 100 cubic numbers. As a challenge, try to do this without a calculator. Factoring helps.

You can do it!

26. What is the sum of the numbers in the 7th row of Pasquale's triangle? The "1" at the top of the triangle is the *zeroth* row, not the first.

Alice still wasn't sure how to reconcile the Mahler polynomials that Kurt introduced and her own work with Pasquale's triangle. But she found it…

27. How would you teach someone to find the number corresponding to $\binom{97}{2}$? Use this to find a formula for $\binom{n}{2}$ as a polynomial.

28. What's the formula for $\binom{n}{3}$? $\binom{n}{4}$? $\binom{n}{k}$?

Alice felt much better about this – apparently she had the same formula all along. She just had to "allow" n to be any number, even though the meaning of $\binom{n}{k}$ expected n to be an integer.

And now, extra for experts...

29. Let $f(x) = x^p$. What's the degree of $\Delta(f)(x)$? What's the degree of $\Delta^2(f)(x)$? ***Prove it!!***

Think about how you would expand on $f(n+1) - f(n)$ where f is a power function.

30. Suppose $f(n) = 2^n$. Find and *prove* the formula for $\Delta(f)(n)$ by using the definition of the Δ operator. How about for $\Delta^2(f)(n)$? $\Delta^k(f)(n)$?

31. Alice really likes that first row, and now wants to bring in her work with the combination numbers. Write $f(5)$ in terms of the combination numbers $\binom{5}{k}$ and the values across the top row: $f(0)$, $\Delta(f)(0)$, and so forth.

32. Find the value of $f(7)$ using the method from problem 31.

33. Write $f(n)$ in terms of combination numbers. Quite a formula, this.

4 *Choose Wisely*

Hey. So we've now found these things called *Mahler polynomials.* $\frac{x(x-1)}{2}$ is referred to as the second Mahler polynomial (since it is of degree 2). The nth Mahler polynomial has roots of 0, 1, 2, 3, ..., $n-1$, and $f(n) = 1$. This helps these polynomials form a *basis*. They can be added to one another very easily to form a polynomial matching a difference table.

1. What is the fourth Mahler polynomial? What about the fifth? Make sure you know how to build the nth Mahler polynomial if you had to.

2. Use Mahler polynomials to find a polynomial with this table. You'll have to take differences first.

Skip this problem if you've already done it...

Input	Output
0	0
1	1
2	5
3	14
4	30
5	55
6	91

3. What is the sum of the numbers in the 7th row of Pasquale's triangle? The "1" at the top of the triangle is the *zeroth* row, not the first.

What? A non sequitir? Surely no...

4. How would you teach someone to find the number corresponding to $\binom{97}{2}$? $\binom{1001}{2}$? Use this to find a formula for $\binom{n}{2}$ as a polynomial. Note that this polynomial can take 5.5 as an input, while $\binom{5.5}{2}$ is not defined.

5. What's the polynomial formula for $\binom{n}{3}$? $\binom{n}{4}$?

So Alice just had to "allow" n to be any number in her "choose" formula, even though the meaning of $\binom{n}{k}$ expects n to be an integer.

6. A suggestion from the crowd gave this one. What happens when you don't start at zero? You can still build the difference table, but it won't be quite the same formula. Say, this table for a quadratic:

Input	Output
3	3
4	-1
5	-1
6	3
7	11
8	23
9	39

What do you do about this? There is more than one good answer here. What could you do if you were just trying to find $f(13)$ and you had this table?

7. What happens when the values don't come in increments of 1? Say, this table for a quadratic:

Input	Output
0	3
5	-1
10	-1
15	3
20	11
25	23
30	39

Now what do you do, hot shot? Try seeing what happens if you replace 5 by k, 10 by $2k$, et cetera.

Why $2k$? Why not?

Alice and Bob are outside at recess when they see their friend Isaac under a tree. Isaac was saying something about deriving the function $x^3 + 3x$, which Alice and Bob assumed must be very easy – they derived this table:

x	$f(x) = x^3 + 3x$	$\Delta f(x)$
0	0	4
1	4	10
2	14	22
3	36	40
4	76	64
5	140	

8. Using any method, find a formula that fits the Δ column.

9. Expand and collect terms for $(x+1)^3 + 3(x+1)$. Why would we ask this question here?

10. Use the definition of Δ to find the formula for the Δ of $f(x) = x^3 + 3x$.

The definition of Δ is $\Delta f(x) = f(x+1) - f(x)$.

11. Find a formula for the Δ of each function.

(a) $f(x) = 3x + 5$ (b) $f(x) = 2x - 7$ (c) $f(x) = ax + b$

Any conjectures?

12. Find a formula for the Δ of each function.

(a) $f(x) = x^2 + 2$ (b) $f(x) = 3x^2 + 5x - 7$ (c) $f(x) = ax^2 + bx + c$

13. Suppose g is a polynomial function of degree m. What can you say about the degree of $\Delta(g)$? What can you say about the degree of $\Delta^2(g)$? At what point will the differences of g become constant? become zero? Do you believe it is true? Good. Prove it if you must.

Hear me now and believe me later...

Meanwhile, Bob was reaping the benefits of this new "differencing" technique. He no longer has to rely on the repetitive (tedious?) use of the "up and over" method to construct a complete table.

14. Let $f(x) = x^3 + 3x$. Find the formulas for $\Delta(f)(x)$, $\Delta^2(f)(x)$, $\Delta^3(f)(x)$, and $\Delta^4(f)(x)$ using the definition of Δ, **rather than creating a table**.

15. Complete the following table.

Notice how the columns go all the way down to the bottom. We have *formulas* now. If the original function is known, so are *all* its Δs.

x	$f(x)$	$\Delta(f)(x)$	$\Delta^2(f)(x)$	$\Delta^3(f)(x)$	$\Delta^4(f)(x)$
0					
1					
2					
3					
4					
5					

16. Find $f(7)$ in terms of the combination numbers $\binom{7}{k}$ and the values across the top row: $f(0)$, $\Delta(f)(0)$, and so forth. Notice how the Δ columns become 0 before the $\binom{7}{k}$ "run out."

17. Find $f(2)$ in terms of the combination numbers $\binom{2}{k}$ and the values across the top row. Notice how the $\binom{2}{k}$ become 0 before the Δs run out.

18. Use Mahler polynomials and the values across the top row of this table to recover the formula that defines f.

Now in her math class, Alice looks up to see her teacher talking about exponential functions. Boy, this class moves from topic to topic way too quickly. A curiosity strikes her…

19. Complete the table for $f(n) = 2^n$.

n	$f(n)$	Δ	Δ^2	Δ^3	$\cdots$
0	1				
1	2				
2	4				
3	8				
4	16				
5	32				
6					
7					

20. Find $f(7)$ in terms of the combination numbers $\binom{7}{k}$ and the values across the top row: $f(0)$, $\Delta(f)(0)$, and so forth. Is there a connection to problem 3?

There are a lot of formulas out there like the one that can be found using problem 20. See what you can find!

We now know that if $f(n) = 2^n$, then $\Delta(f)(n) = 2^n$ as well. What happens when an exponential function has a base other than 2? Let's try and see...

21. Suppose $f(n) = 3^n$.
(a) Describe the difference table for f. In particular, what is a formula for $\Delta^k(f)(0)$?
(b) Express $f(n)$ in terms of the $\Delta^k(f)(0)$.

22. Suppose $f(n) = a^n$.
(a) Describe the difference table for f. In particular, what is a formula for $\Delta^k(f)(0)$?
(b) Express $f(n)$ in terms of the $\Delta^k(f)(0)$.

23. Suppose $f(n) = (1 + a)^n$.
(a) Describe the difference table for f. In particular, what is a formula for $\Delta^k(f)(0)$?
(b) Express $f(n)$ in terms of the $\Delta^k(f)(0)$.

24. Alice was curious about $f(n) = 0^n$, a function that behaves badly at $n = 0$. Alice heard of a function called the *delta function*, with the formula

$$\delta(n) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases}$$

Build a table for this function up to $\delta(7)$ and calculate its differences (yes, there are some). Then, write $\delta(7)$ in terms of combination numbers as you did in problem 32.

Does the delta function's differences still agree with the results from the problems about $f(n) = a^n$?

25. Use the delta function to prove this fact about Pasquale's Triangle: In any row of Pascal's triangle beginning with the first, the sum of even coefficients $\binom{n}{j}$ (where j is even) is equal to the sum of odd coefficients $\binom{n}{k}$ (where k is odd).

26. Use Mahler polynomials to find the sum of the first 100 cubic numbers. As a challenge, try to do this without a calculator. Factoring helps.

You can do it!

As Alice, Bob, Pasquale, Karl, Kurt, and Isaac go home for the holiday, we leave you with some more food for thought (and some fireworks?).

27. Suppose $f(n)$ = the nth Fibonacci number. So, the table of f looks like this:

Any output (after the first two) is the sum of the previous two outputs.

n	$f(n)$
0	0
1	1
2	1
3	2
4	3
5	5
6	8
7	13

(a) Describe the difference table for f. In particular, find a formula for $\Delta^k(f)(0)$ in terms of the f column.
(b) Express $f(n)$ in terms of the $\Delta^k(f)(0)$.

A closed form for this table is difficult to find – it took mathematicians quite a while. We'll find it in Week 3, though, among other things.

28. What if the Fibonacci numbers were *across* as the list of differences at $f(0)$ instead of being the outputs? Can you relate this to the - gasp - numbers in Pascal's triangle using Kurt's notation?

29. Suppose $f(0) = 3$, $f(1) = 6$, $f(3) = 12$, and $f(10) = -2$. Find a cubic polynomial going through these four points. Difference tables won't help you now...

30. **Prove or Disprove and Salvage if Possible**:

To *salvage* means to fix it, then prove it.

f is a constant function if and only if $\Delta(f) = 0$.

31. Time to get ridiculous.

(a) What fraction has decimal expansion 0.020202020202...?
(b) ... decimal expansion 0.466666666666...?
(c) ... decimal expansion 0.538461538461...?
(d) ... 0.461538461538...?
(e) ... 0.010203040506...?
(f) ... 0.020508111417...? (1 less than multiples of 3)
(g) ... 0.010102030508132134...? (Fibonacci)
(h) ... 0.01030927...? (Powers of 3)
(i) ... 0.01082856...? (8th row of Pasquale)
(j) ... 0.0104091625...? (Square numbers)