1 Day 1: The Basics (?)

Hey, welcome to **"The Art and Craft of Adding and Subtracting"**. The intent of these sessions is ostensibly to learn more math, but we really think it's about a teaching philosophy. We don't expect you to get all these problems right; if you can get every problem on every set, we haven't done our job. We are confident that you will learn more math by doing these problems than by us standing at the board at 8:20 am every day for two hours. Hopefully you will agree, and it may help you in your teaching (or, better yet, in others' teaching!). Most of all, have a good time and don't worry about being stuck on a problem. Thanks and enjoy!

It must be important if it's in bold.

Keep on Adding A bored student in a math class started fiddling with the calculator. Alice started with the number 3, then kept on typing +5, +5, +5. Little did she know she was performing *iteration*, or repeated operation, or that her work starts a journey:

1. Make a table, starting with 0, for the number of times Alice performed the +5 operation and the end result of the calculation.

Iterations	Result
0	3
1	
2	
3	
4	
5	
6	

2. Suppose Alice's result when the teacher finally noticed was 188. How many times did she perform the operation? How did you solve this problem?

A "recursive" formula for Alice's operation is

$$f(0) = 3 \qquad f(n+1) = f(n) + 5$$

An alternate version that is equally good is:

$f(n+1) - f(n) = 5.$

This describes what she did to get from one step to the next. The recursive formula isn't that useful, though, if we wanted to find out the value of $f(210)$.

3. Use your work in problem 2 to write a "closed-form" version of Alice's formula. That is, make a formula that says $f(n) = \cdots$ directly.

4. If Alice started with the number a and added b each time, what would the recursive formula be? What would the closed-form formula be?

Later, Alice wrote down the numbers

$$1 \qquad 2 \qquad 3 \qquad 4 \qquad 5$$

then decided to add them in pairs and write the sum underneath. So under those numbers she wrote

$$3 \qquad 5 \qquad 7 \qquad 9$$

... which is the sum of each pair of successive numbers.

5. What's the last number generated this way? Make sure you keep track of this table, you might see it again.

6. In terms of the numbers written in the first row, how is the last number generated? Is there a formula?

Take a few minutes on this.

7. Alice started over, this time with the numbers

$$1 \qquad 10 \qquad 100 \qquad 1000 \qquad 10000$$

What's the last number generated this way? Does it give any insight on a possible formula?

8. Find the last number if Alice started with

$$1 \qquad 2 \qquad 3 \qquad 4 \qquad 5 \qquad 6$$

Can you get it using a formula from the original numbers?
Can you get it using the information from problem 5?

Plenty of food for thought here, and all we did was add!

Keep on Subtracting Bob was presented with a table of information and asked to find the formula. He scoffed at the word "the" and decided to do some other stuff. Here's the table of information:

Input	Output
0	3
1	8
2	13
3	18
4	23
5	28
6	33

9. Find a formula that fits this table. Is this the only formula that fits the table? If yes, explain why; if no, explain why not, and see if you can find another.

Bob (bored or not) decided to take successive outputs and subtract them from one another; this is sometimes referred to as taking the *common difference*.

10. Calculate the common differences for this table.

As a matter of notation, the numbers you generated are the common difference at *zero* through *five*.

11. Write the common difference at zero in terms of the function's inputs. In other words, fill in the blanks on this:

$$d(0) = f(\quad) \qquad f(\quad)$$

12. Is there a formula for the common difference at n in terms of the function's inputs? Is there a relationship to Alice's recursive formula?

And no, "5" is not the answer here.

13. Could Bob have calculated the *second common differences* (differences of the differences)? How far could he go with this?

Bob wondered if this worked the other way, too; if he had the list of common differences, could he figure out what the original numbers were?

14. Suppose Bob knew the common differences are 2, 5, 12, and 28. Is this enough information to tell you what the five outputs were? Suppose Bob also knew the first output was 1. Would that help?

15. Suppose the common differences are 62, 13, 27, and 38 and Bob knows the first output is 10. Calculate the final output as fast as you can.

16. Suppose the common differences are 27, 62, 38, and 13 and Bob knows the first output is 10. Calculate the final output as fast as you can. What's going on here? Can you explain it? Prove it?

Later, Bob received another table:

Input	Output
0	1
1	3
2	8
3	20
4	48

This was a little more interesting, since he could "keep going" with the differences until there was only a single number left.

17. What's the last number generated this way? Make sure you keep track of this table, you might have seen it again.

18. In terms of the numbers written in the second column (i.e., the original outputs), how is the last number generated? Is there a formula?

Take a few minutes on this.

19. Bob started over, this time with the numbers

1 10 100 1000 10000

What's the last number generated this way? Does it give any insight on a possible formula?

Keep on... Both! When Alice and Bob compared notes at lunch, they were shocked and chagrined to find their boredom had led to mathematical insight, and that their work had been so close together. A teacher recommended they organize things this way:

Input	Output	Δ	Δ^2	Δ^3	Δ^4
0	1	2	3	4	5
1	3	5	7	9	
2	8	12	16		
3	20	28			
4	48				

Depending on which way you look, this table is either adding *or* subtracting. If you only know the outputs, you can subtract and determine the differences. The *iteration* is still going on, but this time the thing being iterated is the operation of taking common differences. So, the exponent 2 in Δ^2 doesn't mean "squaring," but repeating the Δ operation twice.

On the other hand, if you only know the first row (the sequence of common differences) you can add your way down and over to find more information. Alice and Bob have both had opportunities to add, and this table includes all that information.

20. On this table, circle the number 16, then shade the two numbers that Alice added to get 16 in problem 5.

21. On this table, circle the number 48, then shade the five numbers, including the "1", that Bob added to get 48 in problem 7.

These will be referred to as the "up and over" and "hockey stick" properties of these tables:

- **Up and over**: Any number in the table (other than inputs) can be calculated as the sum of two numbers – the number directly above it and the number directly to the right of the one above it.

- **Hockey stick**: Any number in the table (other than inputs) can be calculated as the sum of the top value in its column and *all* the numbers above it in the column to its right.

22. Find a function that agrees with this table:

Input	Output	Δ
0	c	d
1		d
2		d
3		d
4		d
5		

23. Find a function that agrees with the table below.

You might want to fill in the table first. Is there only one table that agrees with these "'boundary conditions?"

Input	Output	Δ	Δ^2
0	3	4	2
1			2
2			2
3			2
4			
5			

24. Find a function that agrees with this table:

This one is pretty tough, for now.

Input	Output	Δ	Δ^2
0	a	b	c
1			c
2			c
3			c
4			
5			

25. Suppose you knew that in the table you used in problems 20 and 21 on page 5, the Δ^4 column was *always* "5". Use that information to find $f(10)$ as quickly as possible. Might there be a formula for this?

Boy, wouldn't *that* be nice.

2 *Morning Addition*

In order to keep Alice and Bob a little more busy, the teacher tells them to join Karl in calculating the sum of the first n whole numbers. To make it "easier," she tells them to refer to this function as $f(n)$.

1. What is $f(1)$, the sum of the first 1 whole numbers?

Hint: The answer is *not* 1. (Whole!)

2. Find $f(n)$ for integer n between 0 and 10. Think about what you did while you were filling out the table. Yes, $f(0)$ is zero, since you didn't add anything and zero is the additive identity.

3. Alice figured out a "recursive" formula for this function:

$$f(n+1) = f(n) + \cdots$$

Figure out the missing piece of the formula.

4. Draw a plot of $f(n)$ for n between 0 and 6. What kind of graph is it? What does that imply about the *degree* of the function involved?

Looking up from their work, Bob and Alice saw their teacher introduce the *Factor Theorem*.

Theorem 1 (Factor Theorem) *If $f(x)$ is a polynomial function, and $f(a) = 0$, then $(x-a)$ is a factor of $f(x)$. Put another way, the function can be written as*

$$f(x) = A(x - r_1)(x - r_2)\cdots(x - r_m)$$

where r_1 through r_m are roots *where $f(r_j) = 0$.*

For example, if I know $f(3) = 0$ and $f(5) = 0$, then $(x-3)$ and $(x-5)$ must be factors of the polynomial. There might be others, of course. If I know these are the *only* factors, then the polynomial must be $A(x-3)(x-5)$, and then I just have to find A.

5. For the function Bob and Alice were working on, what are the roots? How do you know? What are the factors, then? Explain how problem 4 suggests that these are the only two factors.

6. Use the information about the roots to find a "closed-form" formula for $f(n)$, then use it to find $f(101)$.

Karl looks on, impressed. He asked about what Alice and Bob were working on yesterday. Bob shows him some problems that not everyone might have gotten to.

7. Suppose Bob knew the first differences are 8, 23, 64, and -7. Is this enough information to tell you what the five outputs were? Suppose Bob also knew the first output was 1. Would that help?

8. Suppose the first differences are 62, 13, -27, and 38 and Bob knows the first output is 10. Calculate the final output as fast as you can.

9. Suppose the first differences are -27, 62, 38, and 13 and Bob knows the first output is 10. Calculate the final output as fast as you can. What's going on here?

10. Build a table of differences, second differences, and third differences for this function $f(n)$. What do you notice about the first differences? The second differences? The third differences? Bob provided Karl with a template.

Input	Output	Δ	Δ^2	Δ^3
0	0			
1	0			
2	1			
3	3			
4	6			
5	10			
6	15			

By the way, the letter Δ stands for "difference" here, and the exponent 2 in Δ^2 doesn't mean "squaring," but repeating the Δ operation twice.

Alice became interested in the rows, rather than the columns

– especially the row with $f(0)$ and its differences. This row was all zeros, except for a 1 as the second difference. More food for thought? A good thing, since lunch was a long way off.

11. Construct a new table where the row with $f(0)$ is all zeros, except for a 5 as the first output. What function is this?

By *all* zeros, we literally mean forever. This means you can keep the table going as long as you want without surprises. Oh, and even though there is more than one answer to the "Which function?" questions, you'll give the right one.

12. Construct a new table where the row with $f(0)$ is all zeros, except for a 1 as the first difference. What function is this?

13. Construct a new table where the row with $f(0)$ is all zeros, except for a 3 as the first difference. What function is this? How is this related to your answer to problem 12?

14. Construct a table where the row with $f(0)$ is all zeros, except the first output is a and the first difference is b. What function is this? How is this related to your answers to problems 11 and 12?

15. Construct a new table where the row with $f(0)$ is all zeros, except for a 2 as the second difference. Use your answer from problem 6 to answer as quickly as possible: What function is this?

16. Imagine a new table where the row with $f(0)$ is all zeros, except for a 1 as the first difference *and* a 2 as the second difference. **Don't construct it!** Instead, predict the function based on your answers to problems 12 and 15 *before* you construct it. Okay, now construct the table. Were you right?

Alice was pretty impressed. She figured she could now predict any function that went all the way up to second differences. This was going to make some of her homework *much* easier.

Remember, Δ refers to the differences, and Δ^2 refers to the second differences.

17. A function has $f(0) = 3$, $\Delta(f)(0) = 4$, $\Delta^2(f)(0) = 2$, and all later differences are zero for all numbers. What is the polynomial function that matches this table? Test your answer by checking your prediction for $f(5)$ against the value you got from the table in yesterday's problem 23. (Didn't do it? Don't worry.) Easier now, ain't it?

So, $\Delta(f)(0) = 4$ means the first difference at zero is 4. Get it?

18. A function has $f(0) = a$, $\Delta(f)(0) = b$, $\Delta^2(f)(0) = c$, and all later differences are zero. What is the function? Don't worry about expanding or collecting terms.

At this, Kurt came by and mentioned that the same idea might be used to figure out the polynomial to use for third differences. This would add to Alice's list of polynomials for everything up to second differences.

19. What happens to the degree of a polynomial each time you perform the "difference" operation? Give some examples. Can you convince yourself it's always true? Can you convince *someone else*?

i.e., prove it?

20. Suppose you knew the second differences were all the same number. What kind of polynomial did you start with?

21. Suppose you knew the third differences were all the same number. What kind of polynomial did you start with? How many *roots* does this kind of polynomial have?

22. Construct a new table where the row with $f(0)$ is all zeros, except for a 1 as the *third* difference. If this is a polynomial fuction, what is its degree? Calculate all the rows out to $f(8)$ and its differences.

23. What are the *roots* of the polynomial in problem 22? How do you know? What are the factors? Use this information to find the function that fits this table. Don't worry about expanding or collecting terms, just leave it factored.

A *root* is a place where the function is zero.

24. A function has $f(0) = a$, $\Delta(f)(0) = b$, $\Delta^2(f)(0) = c$, $\Delta^3(f)(0) = d$, and all later differences are zero. What is the function? Don't worry about expanding or collecting terms, since we are looking for a pattern here.

25. Alice made up a table with $f(0) = 3$, $f(1) = 10$, $f(2) = 23$, and $f(3) = 48$. She was able to find a cubic polynomial to fit this table *very* quickly. Can you?

26. As a joke, Bob added on $f(4) = -5$. Can you find a polynomial to fit this table? How quickly? You'll have to extend the reasoning used in the last few problems, but if you've gotten here, you must be good at that.

Alice and Kurt felt pretty happy about this, and even Karl agreed that this would be a powerful tool. Bob had other ideas, looking at some of the other rows in the table from problem 22. Bob could swear he'd seen some of these numbers before.

Still going? Okay, here's some more.

27. Suppose $f(0) = 1$ and all differences at zero are 1. Yes, forever. What function is this?

28. Suppose $f(n) = 10^n$ for all n. What are its first differences? second differences? Use the definition of the Δ function to explain why this is true.

29. Suppose $f(0) = 1$ and each difference is ten times larger than the one before it. What function is this? Can you prove why? What if each difference is n times larger than the one before it?

cough... *binomial*... cough...

30. Use what you've learned to find the sum of the first 100 fourth powers without calculating them all.

31. Suppose $f(0) = 1$ and each difference is the opposite of the one before it (so it's $1, -1, 1, -1, ...$). How many roots does this function have?

3 *Blazing Through*

It's time to review! Hooray! Take some time and make sure you're able to do this kinda stuff.

1. Find a polynomial function with $f(0) = 5$, $f(1) = -3$, and $f(2) = -11$. Use differences to help!

2. Find a polynomial function with $f(0) = 21$, $\Delta f(0) = -18$, and $\Delta^2 f(0) = 6$.

Alice and Bob hear (but don't particularly listen to) the teacher talking about *combinations* and *permutations*. A combination is a group in which the order does not matter, while in a permutation, order matters. The "Pick-6" style of lottery is a combination. Choosing what shirts to wear over the next five days is usually a permutation problem, unless you really, really like the same shirt.

3. How many ways are there to pick a group of three people out of eight? By "group," it means "you three" and not "you first, you second, then you third." How many ways are there to pick a group of two out of eight? A group of one out of eight? A group of zero out of eight? How about a group of three people out of two?

4. Elbonia has a "Pick-1" lottery, where the player picks a single number and hopes it matches the drawing. How many different possible plays are there in this lottery if there are 49 numbers in the drawing?

5. Freedonia has a "Pick-2" lottery, where the player picks two numbers and hopes they match the drawing (in either order). How many different possible plays are there in this lottery if there are 49 numbers in the drawing? if there are n numbers in the drawing?

The distinction *either order* makes this a combination problem, rather than a permutation problem.

6. Coruscant has a "Pick-3" lottery. Same deal. How many different plays in this lottery for 49 numbers? for n numbers?

The notation $\binom{n}{k}$ is used to represent combinations. Alice and Bob's teacher starts talking about drawn-out formulas for combinations, and as usual, they tune out and resume the stuff they were working on yesterday.

So, the number of plays in a "Pick-6" lottery with 49 numbers is $\binom{49}{6} = 13,983,816$.

7. If you didn't do problem 22 from yesterday, do it now:

Construct a new table where the row with $f(0)$ is all zeros, except for a constant 1 as the third difference for all entries. If this is a polynomial function, what is its degree? Calculate all the rows out to $f(8)$ and its differences.

8. Look at the rows of this table, starting with $f(3)$. Do the four numbers in this row look familiar at all? What about the four numbers in the next row? How about the numbers in the $f(8)$ row?

9. Consider the four numbers in the row starting with $f(3)$. Explain how these numbers are used to generate the row starting with $f(4)$. Does this process seem familar at all?

10. What about the numbers for $f(n)$, from $n = 0$ onward? Do these numbers look familar? How so?

They might not, of course!

Hey, it's fine if you just said, "No, no, NO!!!" to problems 8, 9, and 10. Bob found some patterns, though, so hopefully you did too. Now, for the flip side.

11. **Do this problem by completing the table, even if you know how to get the formula for $f(x)$.** A function has $f(0) = 3$, $\Delta(f)(0) = -4$, $\Delta^2(f)(0) = 5$, and constant $\Delta^3(f)(0) = 2$. Complete the table to find $f(8)$.

The bold thing would've been a side note, but Art would ignore it.

12. Now, explain how $f(8)$ could be calculated from the two numbers directly above it (remember, "up and over"). Write $f(8)$ in terms of these numbers.

13. Now, explain how $f(8)$ could be calculated from the three numbers directly above that. Write $f(8)$ in terms of these numbers, and collect terms if you used the same one more than once. You'll get the most benefit by writing

$$f(8) = m \cdot \_\_\_\_ + n \cdot \_\_\_\_ + \cdots$$

even if some of those m and n are ones.

14. Now, explain how $f(8)$ could be calculated from the *four* numbers directly above that. Write $f(8)$ in terms of these numbers, and collect terms if you used the same ones more than once. Follow the advice in problem 13; it will help to see the overall pattern here.

15. Now, explain how $f(8)$ could be calculated from the *five* numbers directly above *that*. Write $f(8)$ in terms of these numbers, and collect terms if you use the same one more than once (and you will).

What? You only see four? Too bad, there are five – to get the fifth one, you're going to need a bigger table, to paraphrase Roy Scheider.

16. Finally, explain how $f(8)$ could be calculated from *nine* numbers calculated across the top (starting with $f(0)$).

To get the fifth and higher terms, you'll need a bigger table or a pattern from the previous problems.

Alice showed her work to her classmate Pasquale, who really seemed to appreciate these numbers. Pasquale told Alice that if she'd paid attention in class, she might've learned a more convenient way to write the numbers in problem 16 – something to do with a triangle. Whatever.

17. Write $f(8)$ in terms of the combination numbers $\binom{8}{k}$ and the values across the top row: $f(0)$, $\Delta(f)(0)$, and so forth. What does this have to do with Pasquale's triangle?

18. Find the value of $f(10)$ as quickly as possible.

19. How would you find $f(97)$? Find it if you want to, but use a calculator!

20. Find the value of $f(5.5)$ as quickly as possible.

Problem 20 presented a problem (literally) to Alice, since there weren't gaps in the table and there wasn't any way to say how many ways to to pick two people out of a group of 5 and a half. Bob and Kurt felt fine with it, since they have polynomials that they could add to find the answer:

$$f(x) = 3 \cdot 1 + (-4) \cdot x + 5 \cdot \frac{x(x-1)}{2} + 2 \cdot \frac{x(x-1)(x-2)}{6}$$

These polynomials come from yesterday's work. The first difference at zero is -4, the second difference is 5, and so on.

These polynomials (1, x, $\frac{x(x-1)}{2}$, etc.) are called the *Mahler polynomials*. $\frac{x(x-1)}{2}$ is referred to as the second Mahler polynomial (since it is of degree 2). The Nth Mahler polynomial has roots of 0, 1, 2, 3, $\ldots$, $n-1$, and $f(n) = 1$. This helps these polynomials form a *basis*. They can be added to one another very easily to form a polynomial matching a difference table.

21. What is the next (fourth) Mahler polynomial? What about the fifth? What's a formula for the Nth Mahler polynomial?

22. Use Mahler polynomials to find a polynomial with this table. You'll have to take differences first.

Input	Output
0	6
1	5
2	24
3	99
4	290
5	681

23. What is the sum of the first three square numbers? Oh, wait, it's 14. Never mind. What is the sum of the first n square numbers? If $f(n)$ is the sum of the first n square numbers, build a table for $f(n)$ from zero to at least eight.

As before, the sum of the first 0 square numbers is 0, since you haven't added anything.

24. Use Mahler polynomials to find a formula for the sum of the first n square numbers.

25. Use Mahler polynomials to find the sum of the first 100 cubic numbers. As a challenge, try to do this without a calculator. Factoring helps.

You can do it!

26. What is the sum of the numbers in the 7th row of Pasquale's triangle? The "1" at the top of the triangle is the *zeroth* row, not the first.

Alice still wasn't sure how to reconcile the Mahler polynomials that Kurt introduced and her own work with Pasquale's triangle. But she found it…

27. How would you teach someone to find the number corresponding to $\binom{97}{2}$? Use this to find a formula for $\binom{n}{2}$ as a polynomial.

28. What's the formula for $\binom{n}{3}$? $\binom{n}{4}$? $\binom{n}{k}$?

Alice felt much better about this – apparently she had the same formula all along. She just had to "allow" n to be any number, even though the meaning of $\binom{n}{k}$ expected n to be an integer.

And now, extra for experts...

29. Let $f(x) = x^p$. What's the degree of $\Delta(f)(x)$? What's the degree of $\Delta^2(f)(x)$? ***Prove it!!***

Think about how you would expand on $f(n+1) - f(n)$ where f is a power function.

30. Suppose $f(n) = 2^n$. Find and *prove* the formula for $\Delta(f)(n)$ by using the definition of the Δ operator. How about for $\Delta^2(f)(n)$? $\Delta^k(f)(n)$?

31. Alice really likes that first row, and now wants to bring in her work with the combination numbers. Write $f(5)$ in terms of the combination numbers $\binom{5}{k}$ and the values across the top row: $f(0)$, $\Delta(f)(0)$, and so forth.

32. Find the value of $f(7)$ using the method from problem 31.

33. Write $f(n)$ in terms of combination numbers. Quite a formula, this.

4 *Choose Wisely*

Hey. So we've now found these things called *Mahler polynomials.* $\frac{x(x-1)}{2}$ is referred to as the second Mahler polynomial (since it is of degree 2). The nth Mahler polynomial has roots of 0, 1, 2, 3, ..., $n-1$, and $f(n) = 1$. This helps these polynomials form a *basis*. They can be added to one another very easily to form a polynomial matching a difference table.

1. What is the fourth Mahler polynomial? What about the fifth? Make sure you know how to build the nth Mahler polynomial if you had to.

2. Use Mahler polynomials to find a polynomial with this table. You'll have to take differences first.

Skip this problem if you've already done it...

Input	Output
0	0
1	1
2	5
3	14
4	30
5	55
6	91

3. What is the sum of the numbers in the 7th row of Pasquale's triangle? The "1" at the top of the triangle is the *zeroth* row, not the first.

What? A non sequitir? Surely no...

4. How would you teach someone to find the number corresponding to $\binom{97}{2}$? $\binom{1001}{2}$? Use this to find a formula for $\binom{n}{2}$ as a polynomial. Note that this polynomial can take 5.5 as an input, while $\binom{5.5}{2}$ is not defined.

5. What's the polynomial formula for $\binom{n}{3}$? $\binom{n}{4}$?

So Alice just had to "allow" n to be any number in her "choose" formula, even though the meaning of $\binom{n}{k}$ expects n to be an integer.

6. A suggestion from the crowd gave this one. What happens when you don't start at zero? You can still build the difference table, but it won't be quite the same formula. Say, this table for a quadratic:

Input	Output
3	3
4	-1
5	-1
6	3
7	11
8	23
9	39

What do you do about this? There is more than one good answer here. What could you do if you were just trying to find $f(13)$ and you had this table?

7. What happens when the values don't come in increments of 1? Say, this table for a quadratic:

Input	Output
0	3
5	-1
10	-1
15	3
20	11
25	23
30	39

Now what do you do, hot shot? Try seeing what happens if you replace 5 by k, 10 by $2k$, et cetera.

Why $2k$? Why not?

Alice and Bob are outside at recess when they see their friend Isaac under a tree. Isaac was saying something about deriving the function $x^3 + 3x$, which Alice and Bob assumed must be very easy – they derived this table:

x	$f(x) = x^3 + 3x$	$\Delta f(x)$
0	0	4
1	4	10
2	14	22
3	36	40
4	76	64
5	140	

8. Using any method, find a formula that fits the Δ column.

9. Expand and collect terms for $(x+1)^3 + 3(x+1)$. Why would we ask this question here?

10. Use the definition of Δ to find the formula for the Δ of $f(x) = x^3 + 3x$.

The definition of Δ is $\Delta f(x) = f(x+1) - f(x)$.

11. Find a formula for the Δ of each function.

(a) $f(x) = 3x + 5$ (b) $f(x) = 2x - 7$ (c) $f(x) = ax + b$

Any conjectures?

12. Find a formula for the Δ of each function.

(a) $f(x) = x^2 + 2$ (b) $f(x) = 3x^2 + 5x - 7$ (c) $f(x) = ax^2 + bx + c$

13. Suppose g is a polynomial function of degree m. What can you say about the degree of $\Delta(g)$? What can you say about the degree of $\Delta^2(g)$? At what point will the differences of g become constant? become zero? Do you believe it is true? Good. Prove it if you must.

Hear me now and believe me later...

Meanwhile, Bob was reaping the benefits of this new "differencing" technique. He no longer has to rely on the repetitive (tedious?) use of the "up and over" method to construct a complete table.

14. Let $f(x) = x^3 + 3x$. Find the formulas for $\Delta(f)(x)$, $\Delta^2(f)(x)$, $\Delta^3(f)(x)$, and $\Delta^4(f)(x)$ using the definition of Δ, **rather than creating a table**.

15. Complete the following table.

x	$f(x)$	$\Delta(f)(x)$	$\Delta^2(f)(x)$	$\Delta^3(f)(x)$	$\Delta^4(f)(x)$
0					
1					
2					
3					
4					
5					

Notice how the columns go all the way down to the bottom. We have *formulas* now. If the original function is known, so are *all* its Δs.

16. Find $f(7)$ in terms of the combination numbers $\binom{7}{k}$ and the values across the top row: $f(0)$, $\Delta(f)(0)$, and so forth. Notice how the Δ columns become 0 before the $\binom{7}{k}$ "run out."

17. Find $f(2)$ in terms of the combination numbers $\binom{2}{k}$ and the values across the top row. Notice how the $\binom{2}{k}$ become 0 before the Δs run out.

18. Use Mahler polynomials and the values across the top row of this table to recover the formula that defines f.

Now in her math class, Alice looks up to see her teacher talking about exponential functions. Boy, this class moves from topic to topic way too quickly. A curiosity strikes her...

19. Complete the table for $f(n) = 2^n$.

n	$f(n)$	Δ	Δ^2	Δ^3	$\cdots$
0	1				
1	2				
2	4				
3	8				
4	16				
5	32				
6					
7					

20. Find $f(7)$ in terms of the combination numbers $\binom{7}{k}$ and the values across the top row: $f(0)$, $\Delta(f)(0)$, and so forth. Is there a connection to problem 3?

There are a lot of formulas out there like the one that can be found using problem 20. See what you can find!

We now know that if $f(n) = 2^n$, then $\Delta(f)(n) = 2^n$ as well. What happens when an exponential function has a base other than 2? Let's try and see...

21. Suppose $f(n) = 3^n$.
   (a) Describe the difference table for f. In particular, what is a formula for $\Delta^k(f)(0)$?
   (b) Express $f(n)$ in terms of the $\Delta^k(f)(0)$.

22. Suppose $f(n) = a^n$.
   (a) Describe the difference table for f. In particular, what is a formula for $\Delta^k(f)(0)$?
   (b) Express $f(n)$ in terms of the $\Delta^k(f)(0)$.

23. Suppose $f(n) = (1+a)^n$.
   (a) Describe the difference table for f. In particular, what is a formula for $\Delta^k(f)(0)$?
   (b) Express $f(n)$ in terms of the $\Delta^k(f)(0)$.

24. Alice was curious about $f(n) = 0^n$, a function that behaves badly at $n = 0$. Alice heard of a function called the *delta function*, with the formula

$$\delta(n) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases}$$

Build a table for this function up to $\delta(7)$ and calculate its differences (yes, there are some). Then, write $\delta(7)$ in terms of combination numbers as you did in problem 32.

Does the delta function's differences still agree with the results from the problems about $f(n) = a^n$?

25. Use the delta function to prove this fact about Pasquale's Triangle: In any row of Pascal's triangle beginning with the first, the sum of even coefficients $\binom{n}{j}$ (where j is even) is equal to the sum of odd coefficients $\binom{n}{k}$ (where k is odd).

26. Use Mahler polynomials to find the sum of the first 100 cubic numbers. As a challenge, try to do this without a calculator. Factoring helps.

You can do it!

As Alice, Bob, Pasquale, Karl, Kurt, and Isaac go home for the holiday, we leave you with some more food for thought (and some fireworks?).

27. Suppose $f(n)$ = the nth Fibonacci number. So, the table of f looks like this:

Any output (after the first two) is the sum of the previous two outputs.

n	$f(n)$
0	0
1	1
2	1
3	2
4	3
5	5
6	8
7	13

(a) Describe the difference table for f. In particular, find a formula for $\Delta^k(f)(0)$ in terms of the f column.
(b) Express $f(n)$ in terms of the $\Delta^k(f)(0)$.

A closed form for this table is difficult to find – it took mathematicians quite a while. We'll find it in Week 3, though, among other things.

28. What if the Fibonacci numbers were *across* as the list of differences at $f(0)$ instead of being the outputs? Can you relate this to the - gasp - numbers in Pascal's triangle using Kurt's notation?

29. Suppose $f(0) = 3$, $f(1) = 6$, $f(3) = 12$, and $f(10) = -2$. Find a cubic polynomial going through these four points. Difference tables won't help you now...

30. **Prove or Disprove and Salvage if Possible**:

To *salvage* means to fix it, then prove it.

f is a constant function if and only if $\Delta(f) = 0$.

31. Time to get ridiculous.

(a) What fraction has decimal expansion 0.020202020202...?
(b) ... decimal expansion 0.466666666666...?
(c) ... decimal expansion 0.538461538461...?
(d) ... 0.461538461538...?
(e) ... 0.010203040506...?
(f) ... 0.020508111417...? (1 less than multiples of 3)
(g) ... 0.010102030508132134...? (Fibonacci)
(h) ... 0.01030927...? (Powers of 3)
(i) ... 0.01082856...? (8th row of Pasquale)
(j) ... 0.0104091625...? (Square numbers)

Solutions. 2/99. 7/15. 7/13. 6/13. 100/9801. 201/9801 (= 67/3267). 100/9899. 1/97. Terminating. 10100/970299. Yes, we had fun with this. Yes, we used generating functions! Yes, the TI-89 got these answers faster than us.

5 *Delta, delta, delta...*

Hey, hope you had a good weekend. Here are some problems to get you going again.

1. A cubic function has $f(0) = 3$, $f(1) = 10$, $f(2) = 33$, $f(3) = 20$. Calculate $f(8)$ without finding the equation for the cubic.

2. Use the technique of cancellation to find a fraction that equals the infinite sum $1 + 2x + 4x^2 + 8x^3 + 16x^4 + \cdots$. Then, use the formula to find the sum $1 + .02 + .0004 + .000008 + \cdots$.

3. If $f(x) = x^3 - 3x^2$, calculate $\Delta f(x)$ and $\Delta^2 f(x)$.

The definition of Δ is $\Delta f(x) = f(x + 1) - f(x)$.

Isaac tells Alice and Bob that his ideas are stronger than theirs – something about the principal Mathematico, was it – anyway, Alice and Bob decide to take what they've done a little further. They may not be proving much, but a lot of what they do can be proven.

4. Calculate the following differences, smartly. In other words, do it and then think about what it means in general.
   (a) $\Delta(3x^2)$ (b) $\Delta(3x)$ (c) $\Delta(3x^2 + 3x)$ (d) $\Delta(3x^2 + 3x + 3)$

5. Find formulas for the Δ of each function by using the definition of Δ.

   (a) $P(x) = 1$ (b) $Q(x) = x$ (c) $R(x) = x^2$

   (d) $S(x) = x^3$ (e) $T(x) = 5x^3 - 2x^2 + 3x - 1$

6. Consider $f(x) = 2x^2 - x$ and $g(x) = 2x^2 - x + 5$. Compare

Δf and Δg. What's the deal? Explain it.

7. Consider $b(x) = 3x^4 - 2x^3 + 4x^2 - 5x + 3$. Find $\Delta^5 b(x)$. Think before you act... Don't drink and derive.

Thanks to John for this.

8. Suppose that $\Delta f(x) = 4x - 5 = \Delta g(x)$. Are f and g automatically the same function? If so, explain why. If not, are they related in any way?

9. Take $M(x) = \frac{x(x-1)}{2}$. Use the definition of Δ to find $\Delta M(x)$. Is that interesting at all?

10. Take $N(x) = \frac{x(x-1)(x-2)}{6}$. Find $\Delta N(x)$.

Gee, these functions seem familiar somehow...
This function $\binom{x}{k}$ is the kth Mahler polynomial.

11. Alice generated an amazing general formula for

$$\Delta \binom{x}{k} \qquad (k \geq 0)$$

Can you find it? What about

$$\Delta^m \binom{x}{k} \qquad (m \geq 1, k \geq 0)$$

12. Suppose you have a mystery function g with the property that $\Delta g(x) = 5x - 2$. What is...

(a) $\Delta(2 \cdot g(x))$
(b) $\Delta g(x) + x$
(c) $\Delta^2 g(x)$
(d) $\Delta^3 g(x)$

What would you need to know to find a formula for $g(x)$? How close can you get to such a formula?

13. **Prove or Disprove and Salvage if Possible**:

To *salvage* means to fix it, then prove it.

If f is a quadratic, then $\Delta^n f(x)$ is zero for $n \geq 2$.

14. Using the ideas from this section, write a convincing argument to show that, if f is a polynomial and $\Delta^3(f)$ is constant and non-zero, f is a polynomial of degree 3.

15. Use the definition of Δ to prove that $\Delta 2^x = 2^x$.

16. Find all functions $f(x)$ which satisfy:

(a) $\Delta f(x) = f(x)$
(b) $\Delta f(x) = 2f(x)$
(c) $\Delta^2 f(x) - 3\Delta f(x) + 2f(x) = 0$

Would factoring help? This is a tough one, but see what you can do.

17. Suppose $f(x) = 3x^2 + 7$ and $g(x) = 5x - 11$. Calculate $\Delta f(x), \Delta g(x)$, and $\Delta[f(x) \cdot g(x)]$. See if you can use this example to get a general formula for the *Product Rule* for the Δ operator:

$$\Delta[f(x) \cdot g(x)] = \cdots$$

There is also a Product Rule in calculus – it may or may not help!

18. For $g(x) = 5x - 11$, calculate $\Delta g(x), \Delta([g(x)]^2)$, and $\Delta([g(x)]^3)$. Can you generalize this for powers of functions?

This is *not* the same as taking successive Δ's!

19. Some extensions. Try only if you are interested!

(a) Prove the Product Rule using the definition of Δ; that is, $\Delta f(x) = f(x+1) - f(x)$.

(b) Find the *Quotient Rule* for the Δ operator:

$$\Delta \tfrac{f(x)}{g(x)} = \cdots$$

Bob and Alice decide to drop this and head in a completely new direction. They heard about an article in a recent issue of *Mathematics Teacher* magazine, and choose to ignore it completely.

20. What's $2 + i$ multiplied by $4 - 3i$?

21. What is the product of the complex numbers $a + bi$ and $c + di$? Put your answer in (   ) + (   )i form.

22. Generate the formulas for $\cos(x + y)$ and $\sin(x + y)$ by taking the result of problem 21, replacing a with $\cos x$, b with $\sin x$, c with $\cos y$, and d with $\sin y$.

23. Generate the formula for $\cos(nx)$ for each n from 1 to 4. This can be done in several ways, but one way is to keep taking powers of $a + bi$ and looking only at the real part.

24. If $f(x) = \sin x$, find $\Delta f(x)$.

Alice and Bob groan to learn that this wasn't something completely new after all.

25. If $f(x) = \cos x$, find $\Delta f(x)$ and $\Delta^2 f(x)$.

26. Try

27. Hey, while you're at it, find $\Delta f(x)$ for $f(x) = \sin \pi x$, for $f(x) = e^x$, and for $f(x) = \ln x$.

And now, some extra problems... are you reading this? Good for you. Alice and Bob were wondering if anyone cared what they were doing.

28. No matter what function f we have, we know that

$$\Delta f(x) = f(x+1) - f(x)$$

Here, x can be any input to f. If you replace x by 3, you get $\Delta(f)(3) = f(4) - f(3)$. If you replace x by $x+1$, you get $\Delta(f)(x+1) = f(x+2) - f(x+1)$.

Well, $\Delta^2 f = \Delta(\Delta f)$, so, we can apply the above formula to Δf and obtain a formula for $\Delta^2 f$:

This is subtle. It might help to replace Δf in the first two lines by some temporary symbol like g.

$$\begin{aligned}\Delta^2 f(x) &= \Delta(\Delta f)(x)\\ &= \Delta f(x+1) - \Delta f(x)\\ &= (f(x+2) - f(x+1)) - (f(x+1) - f(x))\\ &= f(x+2) - 2f(x+1) + f(x)\end{aligned}$$

(a) Supply a reason for each step in this calculation.
(b) Use this result to find a formula for $\Delta^2 f$ where $f(x) = 5x^2 - 3x + 2$.
(c) Find a formula for $\Delta^3 f$ where $f(x) = 5x^2 - 3x + 2$.

29. Show that, for any function f,

$$\Delta^3(f)(x) = f(x+3) - 3f(x+2) + 3f(x+1) - f(x)$$

Check that this produces the same result as problem 28c for that particular function.

30. Derive a formula for $\Delta^4 f$. Generalize to $\Delta^m f$ for any integer $m \geq 0$.

31. Based on what you've done, explain why if f is a polynomial function, f can be written as

Is this really an "infinite" sum?

$$f(x) = \sum_{k=0}^{\infty} \Delta^k f(0) \binom{x}{k}$$

32. Write each polynomial function as a linear combination of the $\binom{x}{k}$:

The mapping syntax is another way, instead of constantly calling things $f(x)$. Read as "x maps to 1" (the equivalent of $f(x) = 1$).

(a) $x \mapsto 1$ (b) $x \mapsto x$ (c) $x \mapsto x^2$

(d) $x \mapsto x^3$ (e) $x \mapsto x^4$ (f) $x \mapsto x^5$

33. Make a table from the results in problem 32 and see what patterns you can find. Specifically, can you find a way to get the next result from the one (or ones) before it?

6 6-Up: The Un-Δ

It's a fun numerical, then some of the problems from yesterday that Bob and Alice think are especially relevant. Skip any if you already feel comfortable.

1. A six-sided die, uh, number cube has the numbers 1, 1, 1, 1, 2, and 3. Find the probability of rolling a sum total of 4 on two dice. As an extension (if you have time!) try finding the probability of rolling a 4 on three dice, on four dice, on five dice.

2. Find formulas for the Δ of each function by using the definition of Δ.

The definition of Δ is $\Delta f(x) = f(x+1) - f(x)$.

(a) $P(x) = 1$ (b) $Q(x) = x$ (c) $R(x) = x^2$

(d) $S(x) = x^3$ (e) $T(x) = 2x^3 - 4x^2 + x - 18$

3. Make up your own cubic polynomial, then build its full difference table out to $n = 5$. At what point do the differences become constant? At what point do the differences become zero?

4. If $T(x) = 2x^3 - 4x^2 + x - 18$, what is $\Delta^4 T(x)$?

5. Suppose that $\Delta f(x) = 2x - 7 = \Delta g(x)$. Are f and g automatically the same function? If so, explain why. If not, are they related in any way?

6. Show that

$$\Delta \binom{x}{3} = \binom{x}{2}$$

This statement is true for all the Mahler polynomials.

7. For $g(x) = 3x - 7$, calculate $\Delta g(x), \Delta([g(x)]^2)$, and

$\Delta([g(x)]^3)$. Can you generalize this for powers of functions?

For the next few problems, you will need these formulas for sine and cosine:

For all values of α and β,

$$\cos(\alpha+\beta) = \cos\alpha\,\cos\beta - \sin\alpha\,\sin\beta \qquad \text{and,}$$

$$\sin(\alpha+\beta) = \cos\alpha\,\sin\beta + \sin\alpha\,\cos\beta$$

8. If $f(x) = \sin x$, find $\Delta f(x)$.

9. If $f(x) = \cos x$, find $\Delta f(x)$ and $\Delta^2 f(x)$.

10. Hey, while you're at it, find $\Delta f(x)$ for $f(x) = \sin \pi x$, for $f(x) = \sin 2\pi x$, for $f(x) = e^x$.

Karl tried to take what Alice and Bob have done and connect it to the sums of powers.

You might try building a difference table for each one, then using Mahler to get the formula. There is more than one good method here, though.

11. Suppose $f(0) = 0$ and $\Delta f(x) = 1$ for all integer x. Find a formula for $f(x)$.

12. Suppose $f(0) = 0$ and $\Delta f(x) = x$ for all integer x. Find a formula for $f(x)$.

13. Suppose $f(0) = 0$ and $\Delta f(x) = x^2$ for all integer x. Find a formula for $f(x)$.

14. Suppose $f(0) = 0$ and $\Delta f(x) = x^3$ for all integer x. Find a formula for $f(x)$.

15. Suppose $f(0) = 3$ and $\Delta f(x) = 3x^2 + 2x - 5$ for all integer x. Find a formula for $f(x)$.

Can you do this directly, using your work from the last few problems?

Bob and Alice are just about Δ'd out. They notice that the last few problems all talked about knowing the Δ of some function and finding the original. Isaac tells them that deriving things is pretty easy, but doing the opposite is a little harder. He yells at them, "Don't forget the $+C$...", as he starts moving away and continues in that direction.

Bob invents the notation ∇ (pronounced "un-delta", or, if you prefer, "alted") to stand for this operation:

$$g(x) = \nabla f(x) \text{ if and only if } \Delta g(x) = f(x).$$

For example, if $f(x) = 2$, then $\nabla f(x) = 2x + C$. Here, C can be any constant number, and the value of C is determined by initial conditions.

16. For each function below, find its ∇. Do it, then decide what the consequences are.

(a) $\nabla(3x)$ (b) $\nabla(3)$ (c) $\nabla(3x + 3)$
(d) $\nabla(3x^2 + 3x + 1)$ (e) $\nabla(3x^2 + 3x + 4)$

17. Find formulas for the ∇ of each function. You might consider using the results from the last page. Really, you might!

The mapping syntax is another way, instead of constantly calling things $f(x)$. Read as "x maps to 1" (the equivalent of $f(x) = 1$).

(a) $x \mapsto 1$ (b) $x \mapsto x$ (c) $x \mapsto x^2$

(d) $x \mapsto x^3$ (e) $x \mapsto 4x^3 + 6x^2 + 4x + 1$

18. Find a formula for the ∇ of each function as quickly as possible.

(a) $f(x) = 4x^2 + 1$ (b) $g(x) = 8x^2 + 2$
(c) $h(x) = 12x^2 + 3$ (d) $j(x) = 15x^2 + 3x + 5$

19. Suppose that $f(x) = 2x^2 - x$ and $g(x) = 2x^2 - x + 5$. Compare ∇f and ∇g.

20. Suppose that $\nabla f = 4x - 5 = \nabla g$. What can you conclude about f and g? What of the general case?

Bob and Alice work together with Jake, who seems to be an expert on this business about sums of powers. Jake notes that the sum of the first n squares is the same as $\nabla(x^2)$, where the value of C is zero (initial condition). So, the ∇ operation becomes a remarkable way to find formulas for the sums of powers.

21. Write each polynomial function as a linear combination of the $\binom{x}{k}$ (the "Mahler basis"):

(a) $x \mapsto 1$ (b) $x \mapsto x$ (c) $x \mapsto x^2$

(d) $x \mapsto x^3$ (e) $x \mapsto x^4$ (f) $x \mapsto x^5$

This table should benefit you:

$f(x)$	1	$\binom{x}{1} = x$	$\binom{x}{2} = \frac{x(x-1)}{2!}$	$\binom{x}{3} = \frac{x(x-1)(x-2)}{3!}$	$\binom{x}{4}$	$\binom{x}{5}$
1						
x						
x^2						
x^3						
x^4						
x^5						

22. What patterns can you find in the table from problem 21? Chances are you used these patterns to help finish the table; see if you can use patterns only to find the next row.

23. What is the ∇ of the first Mahler polynomial, x? What is the ∇ of the second Mahler polynomial, $\frac{x(x-1)}{2!}$? What is the...? Can you explain why this all is true, in terms of the tables we've been making? How does it help find the sums of powers?

24. Combine the work in problems 21 and 23 to write an expression (in terms of Mahler polynomials) for the sum of the first n squares, the first n cubes, the first n fourth powers, the first n fifth powers.

25. Find the sum of the first 100 fifth powers as quickly as possible.

26. Find the sum of the first 100 tenth powers using this method (you will need to correctly extend the table in problem 21). Jake bragged that he was able to find this answer in $7\frac{1}{2}$ minutes; can you?

Jake actually did this without a calculator, seeing as how they weren't invented in the early 18th century.

Carryovers for experts who finish the first 4 pages too quickly.

Note: This problem isn't right! Its original form seemed very strange. This form is closer to the rule for Integration by Parts but still seems wrong (and we don't know how to fix it right now).

27. Show that if f and g are functions, then

$$\nabla(f(x)\Delta g(x)) = f(x)g(x) - \nabla(g(x)\Delta f(x+1))$$

28. Find an equation whose roots are the squares of the roots of the roots of

$$x^2 - 4x + 1 = 0$$

29. Find an equation whose roots are the 7th powers of the roots of the roots of

$$x^2 - 4x + 1 = 0$$

30. Without multiplying it out, what is the coefficient of x^{18} in

$$(x + x^2 + x^3 + x^4 + x^5 + x^6)^3$$

... of x^{17}? of x^{16}? Think about how you found these answers.

31. Alice, Bob, and Jake each roll a six-sided die, uh, number cube. What is the probability that the sum of the three dice is 18? is 17? is 16? Think about how you found these answers.

You have dealt with recursively defined functions and sequences of numbers before. For example,

$$f(n) = \begin{cases} 1 & \text{if } n = 0 \\ f(n-1) + n^2 + n + 1 & \text{if } n > 1 \end{cases}$$

How about recursively defined sequences of *polynomials*?

For each of the sequences below, write out 6 terms (or more, if you have a CAS) and describe (and establish) some conjectures you see.

32.

$$f(n) = \begin{cases} 1 & \text{if } n = 0 \\ f(n-1) + x & \text{if } n > 0 \end{cases}$$

33.

$$g(n) = \begin{cases} 1 & \text{if } n = 0 \\ x\,g(n-1) + x & \text{if } n > 0 \end{cases}$$

34.

$$h(n) = \begin{cases} 1 & \text{if } n = 0 \\ h(n-1) + x^n & \text{if } n > 0 \end{cases}$$

35.

$$r(n) = \begin{cases} 1 & \text{if } n = 0 \\ x & \text{if } n = 0 \\ r(n-1) + r(n-2) & \text{if } n > 1 \end{cases}$$

36.

$$t(n) = \begin{cases} 1 & \text{if } n = 0 \\ x & \text{if } n = 1 \\ 2x\,t(n-1) - t(n-2) & \text{if } n > 1 \end{cases}$$

7 Circular Reasoning

Alice and Bob's class has moved full-on into trigonometry. Since all of them are tired of working with the Δ (and maybe you agree), they're relatively happy to be working with θ and α and ψ instead.

In trigonometry, angles start from 0 on the horizontal x-axis and increase in the counter-clockwise direction. It's better to use radian measure for angles (why?), so $\frac{\pi}{2}$ radians is vertical. Most problems here will use radians, although the discussion usually starts in degrees.

Where the ray from the origin $(0, 0)$ crosses the unit circle determines cosine and sine. The cosine value is the x-coordinate of the intersection, and the sine value is the y-coordinate. These can also be thought of as a single number in the complex plane, using the vertical as imaginary: $x + yi$, or $\cos\theta + i\sin\theta$.

1. Draw a large unit circle, and mark in Quadrant I (the upper right) a random point on the circle. Suppose you knew its coordinates and angle $36.9^o : (\frac{4}{5}, \frac{3}{5})$. Mark *seven* other points on the circle whose coordinates and angles are related. For example, one is $(36.9 + 180)^o : (-\frac{4}{5}, -\frac{3}{5})$.

2. What about negative angles? How do the cosine and sine of $(-\theta)$ relate to the cosine and sine of θ? You might draw a circle to help.

3. Suppose $z = x + yi$ is a point on the unit circle. The *conjugate* of a complex number is defined as $\overline{z} = x - yi$. How are the angles corresponding to these two related? Multiply z and $\overline{z}$ to show that $z\overline{z} = 1$.

4. Suppose $z = \frac{4}{5} + \frac{3}{5}i$. Calculate z, z^2, z^3, z^4, and z^5 and then plot each of these powers on a unit circle. What happens to the angle each time (it's okay to use degrees here)? Does this give any insight into why angles in trigonometry are measured counterclockwise, starting from the horizontal x-axis?

5. Suppose $z = \frac{4}{5} + \frac{3}{5}i$ and $w = \frac{12}{13} + \frac{5}{13}i$. Calculate zw and plot z, w, and zw on a unit circle. What happens to the angles involved?

6. Suppose $z = 2 + i$ and $w = 3 + 4i$. Calculate zw and plot z, w, and zw in the complex plane (they won't fit on the unit circle). What happens to the length of zw? What happens to the angles involved if they are measured in the standard trig way?

7. Suppose $z = \cos\theta + i\sin\theta$ and $w = \cos\phi + i\sin\phi$ are points on the unit circle. Calculate zw by using complex number arithmetic.

8. Use the results of problem 7 to extract the formulas for $\cos(\theta + \phi)$ and $\sin(\theta + \phi)$. If you need to see the formulas, they are on yesterday's set.

9. Use one of the formulas from problem 8 to prove the Pythagorean identity:

For any value of θ,

$$\sin^2\theta + \cos^2\theta = 1$$

Your work in problem 3 might come in handy.

10. An extension for interested parties: Use the methods developed here to find the formula for $\cos(\theta + \phi + \alpha)$, and to establish the cofunction identity $\cos(\theta) = \sin(\frac{\pi}{2} - \theta)$.

Alice and Bob really like the angle-sum formula (well, maybe), and hope to see it again Real Soon Now. Seriously, though, having it connect to other branches of mathematics make Alice and Bob appreciate it more than if it had been proved using the distance formula on the unit circle.

But wait, there's more!

11. Draw a unit circle. Mark in one color all the places on the unit circle that have $\cos\theta$ equal to zero, and mark in a second color all the places that have $\sin\theta$ equal to zero.

12. For what angles θ is $\cos 2\theta$ equal to zero? What about $\sin 2\theta = 0$? Extend to 3θ.

13. Draw a unit circle, and mark in one color all the places where $\cos 2\theta$ equals zero. What angles are these? Use those angles and happy special triangle formulas to find the value of $\cos\theta$ at each of the points you marked.

14. How many different values of $\cos\theta$ did you find in problem 13? Suppose these values are the only two *roots* of a polynomial. Find a "nice" polynomial with these roots. By "nice" we mean positive leading coefficient, all integer coefficients, and no common factors between them.

So, $3x^2 - 2x + 3$ is "nice" but $-3x^2 + 2x - 3$ and $x^2 - \frac{2}{3}x + 1$ would not be.

Help! One way to do this is to determine the sum and product of the roots first; then, the polynomial has the form $p(x) = A(x^2 - bx + c)$, where b is the sum of the roots and c is the product. Another way is to write the polynomial as $p(x) = A(x - r_1)(x - r_2)$. Either one will work here.

15. Draw a unit circle, and mark in one color all the places where $\cos 3\theta$ equals zero. What angles are these? Use those angles and different happy special triangle formulas to find the value of $\cos\theta$ at each of the points you marked.

16. How many different values of $\cos\theta$ did you find in problem 15? Suppose these values are the only *roots* of a polynomial. Find a "nice" polynomial with these roots. The calculator might help you with the multiplying out of all them free radicals.

17. Another extension for interested parties only. Try repeating this problem for $\cos 4\theta$ if you like. You will need experience with the half-angle formula $\cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$ to find the values of cosine. Skip this problem if you like; we will find a better way to get these polynomials...

Alice and Bob show the complex number multiplication formula to Abraham, and he says that it might be interesting to look at multiplying the same complex number again and again, as in problem 4. Looks like there might some interesting stuff here.

18. Take the complex number $x + yi$ and calculate its square in the form

$$(\qquad) + (\qquad)i$$

19. If $z = 2 + i$, calculate z^2 and plot both z and z^2. What happens to the angle when you square a complex number?

20. Replace x with $\cos\theta$ and y with $\sin\theta$ in problem 18 to get the formulas for $\cos 2\theta$ and $\sin 2\theta$.

21. Use the rule $\sin^2\theta = 1 - \cos^2\theta$ to rewrite the rule for $\cos 2\theta$ using only cosines. Seen this around anywhere?

22. Take the complex number $x + yi$ and cube it to find the formula for $\cos 3\theta$. Then, use the rule $\sin^2\theta = 1 - \cos^2\theta$ to rewrite the rule using only cosines. Seen this around?

23. Find a formula for $\cos 4\theta$ that has only cosines. Can you find the roots of this polynomial?

By the way, in general we have DeMoivre's Theorem:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

This can be proven for positive integer n using the proof of complex multiplication involving similar triangles. It works for any real number n, too, but the proof requires Taylor series and an important connection between the Taylor series for sine, cosine, and e^x:

$$\begin{aligned} e^x &= 1 + x + \tfrac{x^2}{2!} + \tfrac{x^3}{3!} + \tfrac{x^4}{4!} + \tfrac{x^5}{5!} + \cdots \\ \sin x &= x - \tfrac{x^3}{3!} + \tfrac{x^5}{5!} - \cdots \qquad \text{(only when } x \text{ is in radians!)} \\ \cos x &= 1 - \tfrac{x^2}{2!} + \tfrac{x^4}{4!} - \cdots \qquad \text{(only when } x \text{ is in radians!)} \end{aligned}$$

This leads to the remarkable result that

$$e^{i\theta} = \cos\theta + i\sin\theta$$

and in particular, if you let $\theta = \pi$, $e^{i\pi} = -1$. Amazing but true, and it's the exponential that can be used to prove DeMoivre's theorem in general.

8 *Multiple Ang-xiety*

Alice and Bob are introduced to a new exchange student, Pafnuty. He says that his teacher still can't spell his last name correctly. No one seems to agree how to spell it.

Still, he has some valuable contributions to Alice and Bob's work.

1. Review what you did yesterday on problems 18 through 22 from yesterday. (If the class or groups didn't get there, consider it Problem Set 7 $\frac{1}{2}$.)

2. Write the formula for $\cos 2\theta$ in terms of cosines only. Do the same for the formula for $\cos 3\theta$.

3. Write the formula for $\cos 1\theta$ in terms of cosines only. Yes, this is a relatively simple problem!

4. Write the formula for $\cos 0\theta$ as simply as possible. Test some values of θ if you're not sure. This should be a *very, very* simple formula.

5. Write the formulas for $\cos n\theta$ for $n = 0, 1, 2, 3$ in order. Do you see any patterns to the formulas? See if you could have predicted the formula for $\cos 3\theta$ from the others.

Help! The pattern might be easier to find if you replace all the "$\cos\theta$" terms with a variable (say, t). There's a recursive formula here that involves the previous two polynomials.

6. (a) Try predicting the formula for $\cos 4\theta$ from the previous formulas.

(b) Calculate $(x+yi)^4$ as a number in the form

$$(\quad\quad) + (\quad\quad)i$$

(c) Take the real part of $(x+yi)^4$ to write a formula for $\cos 4\theta$ in terms of both sines and cosines.

(d) Use the substitution $\sin^2\theta = 1 - \cos^2\theta$ to write a formula for $\cos 4\theta$ in terms of cosines only.

7. Find the formula for $\cos 5\theta$ any way you like. If you've found the recursion, that is probably the simplest way to do it. Test your formula by plugging the value $\frac{1}{2}$ in for $\cos\theta$ (or t if you prefer) and evaluating. If you have the right formula, the result should be $\frac{1}{2}$. In other words, if $\cos\theta$ is $\frac{1}{2}$, then $\cos 5\theta$ is also $\frac{1}{2}$. (It's a 60^o angle.)

8. For what angles is $\cos 5\theta$ equal to zero? For what angles is $\sin 5\theta$ equal to zero?

9. Find the formula for $\cos 6\theta$ any way you like. Test your formula by plugging in the value $\frac{1}{2}$ in for $\cos\theta$ and evaluating. If you have the right formula, the result should be 1.

10. Use the recursion to find the seventh and eighth polynomials. If you haven't found the recursion yet, look at the polynomials you've built and see if you can track it down.

One interesting sidelight here is to try and find a way to write exact (the technical term is *algebraic*) forms for sines and cosines of angles. The normal angles students learn are multiples of 30^o and 45^o. Once the angle-sum formulas are known, the ratios for multiples of 15^o can be found. The point, though, is that all these answers can be written in exact form: for example, $\sin 15^o = \frac{\sqrt{6}-\sqrt{2}}{4}$. Some angles cannot be written in this way: for example, there is no exact answer for $\sin 20^o$.

A geometric consequence is that only some algebraic numbers (those involving square roots) can be constructed with straight edge and compass. So, for example, a regular polygon with 150^o angles can be created using only standard geometric construction tools, but a regular polygon with 160^o angles cannot, and this is for the same reason that we can get an exact value for $\sin 30^o$ but no exact value for $\sin 20^o$ exists.

So, the next problems focus on angles that are multiples of 18^o. Alice found that if she took a regular pentagon and connected the diagonals, an interesting triangle shows up inside.

11. There's only one isosceles triangle (up to similarity) whose base angle is twice the vertex angle. It is the "72-72-36 triangle."

For geometry teachers: One consequence of this problem is that the side of a regular decagon inscribed in a circle can be constructed with straight-edge and compass. So the regular decagon, 20-gon, and 10n-gon are all constructible with these tools. So is the pentagon (connect every other vertex of the decagon). How long is the side of the pentagon if the circle's radius is taken to be 1?

Suppose the equal sides of the triangle have length 1, and let q stand for the length of the base.

(a) Bisect one of the base angles of the triangle.

(b) Show that the small triangle is similar to the whole triangle.

(c) Use question 11b to show that

$$\frac{1}{q} = \frac{q}{1-q}$$

(d) Show that $\frac{q}{2} = \cos 72^o$.

(e) Solve for q and for $\cos 72°$. Did you get an exact answer?

12. An extension for those who like radicals: Use the results of problem 11 to find the exact value (in terms of radicals) of $\cos 18^o$. You won't need to build another triangle; use the two relationships between sine and cosine:

$$\sin^2\theta + \cos^2\theta = 1$$

and

$$\sin\theta = \cos(90^o - \theta)$$

13. Non sequitir time... suppose function g has this formula:

$$g(n) = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

Calculate $g(8), g(9)$, and $g(10)$. Interesting?

Interesting to see the connection between the numbers used in the formula in problem 13 and the values of sine and cosine. (We hope.)

So, you should now have all these polynomials representing $\cos n\theta$ for $n = 0, 1, 2, 3, \ldots$. We're going to look at these as polynomials in the variable x (instead of $\cos\theta$), so the polynomial for $\cos 2\theta$ is $2x^2 - 1$. And so on. This polynomial $(2x^2 - 1)$ is referred to as the *second Chebyshev polynomial* and the notation for it is $t_2(x)$.

14. Plot the fourth Chebyshev polynomial on the domain $-10 \leq x \leq 10$. What part(s) of this domain are "interesting"? Can you explain why, knowing what x really represents here?

15. Plot each Chebyshev polynomial, separately, on the interesting domain, starting with the zeroth ($t_0(x) = 1$), then the first ($t_1(x) = x$), then the second ($t_2(x) = 2x^2 - 1$), and out to the eighth. Do you notice any patterns of behavior among these plots?

16. (a) What is the definition of an *even* function?
(b) What is the definition of an *odd* function?
(c) What kind of function do you get when you add two even functions? when you add two odd functions? **Be careful!**
(d) What kind of function do you get when you multiply an even function and an odd function? two odd functions?
(e) Which Chebyshev polynomials are even? Which are odd? Which are neither?
(f) An extension: Look back on the recursive pattern you found. Can you use it to *prove* which Chebyshev polynomials will be even or odd?

17. Do the Chebyshev polynomials ever share roots? Give as many examples as you can find (it may help to plot the polynomials on the same graph).

18. Factor each Chebyshev polynomial (our brand new TI-89's may be of good use here). How do the factorings relate to the results in problem 17?

19. What patterns do you see in the coefficients of these polynomials?

Leftovers.

20. Suppose f is a function on the complex numbers that adds $3-2i$ to everything:

$$f(w) = w + (3 - 2i)$$

Find:
(a) $f(1+5i)$ (b) $f(-4+6i)$ (c) $f(1)$ (d) $f(i)$

21. Consider (again) the function $f(w) = w+z$ with $z = 3-2i$.
(a) Plot w and $f(w)$ for 3 values of w.
(b) Describe how to get $f(w)$ from w, geometrically.

You choose the values of w. Not the ones used in problem 20!

22. Show that if z and w are complex numbers, the distance between z and w on the complex plane is $|z-w|$.

23. Write each complex number as a product of a positive real number and a complex number of length 1:
(a) $4+3i$ (b) $5+12i$ (c) $1+i$
(d) $1-i$ (e) $2+3i$ (f) $-1+i\sqrt{3}$

Note: A complex number of length 1 is just a complex number whose absolute value is 1. (So, the length of its vector is 1.)

24. Show that any non-zero complex number can be written as the product of a positive real number and a complex number of length 1.

25. Use the complex numbers $z = 2+2i$ and $w = -1-2i$ to do the following.
(a) Plot z and w
(b) Find $z+w$ and $|z+w|$
(c) Plot $z+w$
(d) Find $|z|$, $|w|$, $|z|+|w|$
(e) Which is bigger, $|z|+|w|$ or $|z+w|$?

26. Use the picture below to explain the following theorem:

This theorem is sometimes called the "triangle inequality."

Theorem 2

If z and w are complex numbers,

$$|z+w| \le |z| + |w|$$

Use geometry, not algebra, to explain what the theorem says.

27. (a) Find 3 pairs of complex numbers such that $|z+w| = |z| + |w|$.

(b) Generalize to describe all pairs of complex numbers in which:

$$|z+w| = |z| + |w|$$

28. Let $\zeta = \dfrac{1+i\sqrt{3}}{2}$. Show that the powers of ζ lie on the vertices of a regular polygon in the complex plane.

Refer to the above example for these problems.

You'll get much more out of the above figure if you create it yourself. Start with the five points on the small circle, multiply each by $\sqrt{2}-\sqrt{2}i$, and plot the points you get.

29. (a) Why does multiplying by $\sqrt{2}-\sqrt{2}i$ move the points to the circle of radius 10, too?

(b) If you multiplied every point on the circle of radius 5 by $\sqrt{2}-\sqrt{2}i$, would the results lie on the circle of radius 10? Why or why not?

(c) Find four more complex numbers that would also move the points to circle of radius 10, under multiplication.

30. As you would expect, multiplying by 2 is not the same as multiplying by $\sqrt{2}-\sqrt{2}i$. What differences do you notice?

31. (a) Choose five points on the circle of radius 1 and graph them.

(b) Multiply the points by 2 and graph the results. Describe how multiplying by 2 affects the location of the points.

(c) Multiply the points by $\sqrt{2}-\sqrt{2}i$ and graph the results. Describe how multiplying by $\sqrt{2}-\sqrt{2}i$ affects the location of the points.

32. Choose five points on the circle of radius 3, and repeat steps (b) and (c) of problem 31. Make a general conjecture about how multiplying by 2 and multiplying by $\sqrt{2}-\sqrt{2}i$ affect the locations of points in the complex plane.

33. Describe (as completely as you can) how multiplying by z affects the locations of points in the complex plane.

(a) $z = \frac{1}{4}$ (b) $z = -1$
(c) $z = 3$ (d) $z = -5$
(e) $z = i$ (f) $z = -i$
(g) $z = -3i$ (h) $z = 1 + i$
(i) $z = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ (j) $z = \sqrt{3} - i$

You may want to choose some points on the circle of radius 1, as you did in problem 31, to form conjectures.

34. What complex number (in $a + bi$ form) do you get if you rotate $2 + 3i$ through an angle of

(a) $90°$ (b) $60°$ (c) $180°$ (d) $215°$
(e) $150°$ (f) $-45°$ (g) $390°$

35. Find two "nice" complex numbers whose product is

(a) i (b) $\cos 75° + i \sin 75°$ (c) -1
(d) $\cos 165° + i \sin 165°$ (e) $\cos 105° + i \sin 105°$ (f) $\frac{1}{2} + \frac{\sqrt{3}}{2}i$

36. Plot the first 10 powers of each complex number, starting with $z^0 = 1$ (the zeroth power)

(a) $z = i$ (b) $z = 2i$
(c) $z = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ (d) $z = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$
(e) $z = 1 + i$ (f) $z = \frac{3}{10} - \frac{3}{10}i$
(g) $z = -\frac{3}{5} - \frac{4}{5}i$ (h) $z = 3 + 4i$
(i) $z = \cos 20° + i \sin 20°$ (j) $z = -\frac{1}{4} - \frac{1}{3}i$
(k) $z = \cos 72° + i \sin 72°$
(l) $z = \cos(-36°) + i \sin(-36°)$

37. Describe the picture you get when you plot the powers of a complex number z $(1, z, z^2, z^3, z^4, \ldots)$ if:

(a) $|z| = 1$ (b) $|z| > 1$ (c) $|z| < 1$

38. For $z = \cos 72° + i \sin 72°$, find

(a) $\arg(z^2)$ (b) $\arg(z^3)$ (c) $\arg(z^5)$ (d) $\arg(z^{10})$ (e) $\arg(z^n)$

39. The powers of $\cos 30° + i \sin 30°$ lie on a regular polygon in the complex plane. How many sides does this polygon have?

40. Find a complex number whose powers (starting with the zeroth power) form the vertices of a regular polygon with n sides.

(a) $n = 3$ (b) $n = 5$ (c) $n = 7$ (d) $n = 4$ (e) $n = 6$

41. Here's a picture of a regular octagon inscribed in the unit circle on the complex plane. The eight vertices are complex numbers. One of these vertices is $1 + 0i$. Find the others.

Can you write your answers without using sines and cosines?

Show that all the vertices are powers of one vertex.

42. The graph below shows the plot of the powers (from 0 to 21) of
$z = \cos 17° + i \sin 17°$.

z^{21} doesn't make it. z^{22} would be back in the first quadrant.

(a) Why aren't these powers of z vertices of a regular polygon?

(b) Will the powers of z ever come back to 1? If so, how many times around will it take? If not, why?

(c) Describe all θs for which the powers of $z = \cos\theta + i\sin\theta$ are vertices of a regular polygon.

43. Show that if $z = \cos 72° + i \sin 72°$:

(a) $\zeta^4 = \overline{\zeta} = \dfrac{1}{\zeta}$ (b) $\zeta^3 = \overline{\zeta^2} = \dfrac{1}{\zeta^2}$ (c) $\zeta + \zeta^4 = 2\cos 72°$

44. Consider our equation $x^5 - 1 = 0$

(a) Factor $x^5 - 1$.

(b) Show $\zeta^4 + \zeta^3 + \zeta^2 + \zeta = -1$.

45. Let $a = \zeta + \zeta^4$ and $b = \zeta^2 + \zeta^3$.

(a) Show $a + b = -1$. (b) Show that $ab = -1$.

46. Problem 45 says that a and b are two numbers whose sum is -1 and whose product is -1.

(a) Show that they are therefore roots of the equation

$$x^2 + x - 1 = 0$$

(b) Use this equation to find a and b.

47. (a) Show that:

$$\cos 72° = \frac{-1 + \sqrt{5}}{4}$$

(b) What is $\sin 72°$ (exactly)?

48. Write algebraic expressions for each of the roots of $x^5 - 1 = 0$ that do not involve cosine and sine.

Rest of problems from original list.

49. Show that the degree of the Chebyshev polynomial t_k is k. Find a formula for the coefficient for x^k in t_k.

50. Find a formula for the coefficient of some term in t_k other than the highest degree one, x^k.

51. State and prove a fact about Chebyshev polynomials.

52. Express $\cos\left(\frac{\theta}{2}\right)$ in terms of $\cos\theta$

53. Find a expression for $\cos 18°$ and $\cos 54°$ that involves only the four operations of arithmetic and square roots.

54. Find a expression for $\cos 72° + i \sin 72°$ that involves only the four operations of arithmetic and square roots. Use this to express the roots of $x^5 - 1 = 0$ without any sines and cosines.

55. Use a CAS or graphing utility to graph the first 6 or 7 Chebyshev polynomials. Write down any observations about patterns that you see.

You can also use the CAS to find the zeros of these polynomials.

56. Use factoring to solve for the zeros of $t_n(x)$, up to $n = 5$. Do you notice anything about these zeros?

57. Using trig and algebraic techiques, now find the zeros of $t_n(\cos\theta)$, also up to $n = 5$. Does finding the zeros of these polynomials help you to find the zeros of the polynomials from problem 56?

58. It's rumored that there's an explicit formula for t_k:

$$t_k(x) \overset{??}{=} \sum_{j=0}^{\infty} (-1)^j \binom{k}{2j} x^{k-2j}(1-x^2)^j$$

Write out (or use a CAS to calculate) the first few t_k (according to this formula) to see if they really are the Chebyshev polynomials. Is the rumor true? If so, prove it. If not, find an example where it fails.

59. Develop a sequence of polynomials u_k so that

$$\sin k\theta = \sin\theta \cdot u_k(\cos\theta)$$

When is $\sin k\theta$ simply a polynomial in $\sin\theta$?

60. It's rumored that there's another explicit formula for t_k:

$$t_k(x) = \frac{(x+\sqrt{x^2-1})^n + (x-\sqrt{x^2-1})^n}{2}$$

How could such a rumor have gotten started? Is it true?

Week 3

Problems for Day 1.

61. What is the distribution of sums on 2 dice? What is (are) the most common sum(s)?

62. How about three dice? What is (are) the most common sum(s)?

- This would get increasingly difficult for four dice, but check this out: we can use polynomials to get the same distributions:
  - For 2 dice:

$$(x + x^2 + x^3 + x^4 + x^5 + x^6)^2$$

  - For 3 dice:

Just use your CAS.

$$(x + x^2 + x^3 + x^4 + x^5 + x^6)^3$$

63. What is the distribution of sums on 4 dice, 5 dice?

64. If one die is fair and the other is weighted $(1, 1, 3, 4, 5, 6)$, what is the new distribution?

65. If two dice are fair and another is weighted $(1, 1, 3, 4, 5, 6)$, what is the new distribution? What sums do we not get here that we got with 3 fair dice.

66. What is the distribution on three dies: $(0, 2, 3, 4, 5, 5)$,$(0, 1, 1, 2, 2, 2)$, and $(1, 2, 3, 6, 6, 6)$?

67. In general, what do the coefficients in the polynomials represent? What do the exponents in the polynomials represent?

68. Extra: On 2 dice, labeled with any combination of 0-6, how would the dice have to be labeled so that the sums 1-12 are equally likely to come up?

Problems for Day 1/Day 2.

69. How many ways can you make 30 cents with 5 and 8 cent stamps?

Can't you use the polynomials like in the dice problem?

70. How many ways can you make 40,52, 60 cents with 5 and 8 cent stamps?

71. What is the largest number you can't make with 5 and 8 cent stamps?

72. What numbers can you make if you had six 5 cent stamps and ten 8 cent stamps?

73. What can you make with 5, 8, and 11 cent stamps?

74. What can you make if you had unlimited 5 and 8 cent stamps, but only one 11 cent stamp? What if you had two 11 cent stamps?

75. What can you make if you only have twenty 5 cent stamps and thirty 8 cent stamps?

76. How many ways can you make change for a dollar with pennies and nickels?

77. How many ways can you make change for a dollar with pennies, nickels, dimes, quarters and half dollars?

Problems for Day 2 /Day 3

78. Write $0.\overline{142857}$ as a fraction.

79. Write $0.\overline{351}$ as a fraction.

80. Can you write $0.\overline{39024}$ as a fraction without a calculator.

EXAMPLE

To write $0.\overline{384615}$ as a fraction, think of it as an infinite series:

$$\begin{aligned} S &= 3\cdot 10^{-1}+8\cdot 10^{-2}+4\cdot 10^{-3}+6\cdot 10^{-4}+1\cdot 10^{-5}+5\cdot 10^{-6} \\ &\quad +3\cdot 10^{-1}+8\cdot 10^{-2}+4\cdot 10^{-3}+6\cdot 10^{-4}+1\cdot 10^{-5}+5\cdot 10^{-6}+\ldots \end{aligned}$$

Multiply by 10 to the power of the period (in this case six). Thus:

$$\begin{aligned} 1000000S &= 3\cdot 10^5 + 8\cdot 10^4 + 4\cdot 10^3 + 6\cdot 10^2 + 1\cdot 10^1 + 5\cdot 10^0 \\ &\quad +3\cdot 10^{-1} + 8\cdot 10^{-2} + 4\cdot 10^{-3} + 6\cdot 10^{-4} + 1\cdot 10^{-5} + 5\cdot 10^{-6} + \ldots \end{aligned}$$

which is

$$1000000S = 384615+3\cdot 10^{-1}+8\cdot 10^{-2}+4\cdot 10^{-3}+6\cdot 10^{-4}+1\cdot 10^{-5}+5\cdot 10^{-6}+\ldots$$

Now subtract:

$$\begin{aligned} 1000000S &= 384615 + 3\cdot 10^{-1} + 8\cdot 10^{-2} + 4\cdot 10^{-3} + 6\cdot 10^{-4} + 1\cdot 10^{-5} + 5\cdot 10^{-6} + \ldots \\ -S &= 3\cdot 10^{-1} + 8\cdot 10^{-2} + 4\cdot 10^{-3} + 6\cdot 10^{-4} + 1\cdot 10^{-5} + 5\cdot 10^{-6} + \ldots \\ \\ 999999S &= 384615 \end{aligned}$$

So

$$S = \frac{384615}{999999} = \frac{5}{13}$$

By doing operations on the infinite series, we are able to gain a fractional representation for any repeating decimal.

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81. Here are some more to write as fractions:

(a) $0.\overline{63}$
(b) $1.\overline{54}$
(c) $0.35\overline{714275}$
(d) $0.\overline{270}$

82. Try summing:

$$\begin{aligned} &0.1+ \\ &0.02+ \\ &0.003+ \\ &0.0004+ \\ &\vdots \end{aligned}$$

Can you write it as a fraction? When you get to two digit numbers, just make sure the last digit is one to the right of the previous last digit. For example:

$$\begin{aligned} &\vdots \\ &0.000000009+ \\ &0.0000000010+ \\ &0.00000000011+ \\ &\vdots \end{aligned}$$

The infinite sum in problem 82 can be written in summation notation:

$$\sum_{n=1}^{\infty} n\cdot 10^{-n}$$

83. Write out some of these infinite sums and then write them as a fractions:

(a)

$$\sum_{n=1}^{\infty}(2n-1)\cdot 10^{-n}$$

(b)

$$\sum_{n=1}^{\infty} 2n\cdot 10^{-n}$$

(c)

$$\sum_{n=1}^{\infty} 9n\cdot 10^{-n}$$

(d)

$$\sum_{n=1}^{\infty} 6n\cdot 10^{-n}$$

84. Compare the fraction for problem 82 to those in problem 83. Compare83a to 83c and 83d. Compare 83b to 83d. What do you see?

85. The geometric series S is given by:

$$S = 1 + x + x^2 + x^3 + x^4 + \dots$$

Play with the geometric series by multiplying or dividing the whole polynomial and the subtracting parts of it as you did in the previous exercises until you obtain the familiar

$$S = \frac{1}{1-x}$$

Thus, the generating function of the geometric series is

$$F(x) = \frac{1}{1-x}$$

86. Find $F(x)$ for the geometric series when $x = \frac{1}{2}$, when$x = \frac{2}{3}$, or $x = \frac{\pi}{4}$.

Problems for Day 3/4. The Fibonacci sequence is the sequence of integers that starts of 0, 1 and has the property that any number after the first two is the sum of the two before it. Here are some terms:

F_0	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	...
0	1	1	2	3	5	8	13	21	34	...

It turns out that, as n approaches infinity, the ratio

$$\frac{F_{n+1}}{F_n}$$

approaches the golden ratio

$$\frac{1+\sqrt{5}}{2}$$

The exercises below may help us see why that is.

Much as we did for the geometric series, we can find a generating function for the Fibonacci numbers (we use $G(x)$ as the instead of the $F(x)$ we used in the geometric series to avoid unnecessary confusion):

$$G(x) = \sum n = 0^{\infty} F_n x^n$$

Because

$$F_{n+1} = F_n + F_{n-1}$$

we have

$$F_2x + F_3x^2 + F_4x^3 + \ldots = \Big\{F_1x + F_2x^2 + F_3x^3 + \ldots\Big\} + \Big\{F_0x + F_1x^2 + F_2x^3 + \ldots\Big\} \qquad (*)$$

87. Show that $(*)$ becomes:

$$\frac{G(x)}{x} - 1 = G(x) + xG(x) \qquad (**)$$

88. Solve $(**)$ for $G(x)$, showing that

$$G(x) = \frac{x}{1-x-x^2}$$

3) Factor the denominator of $G(x)$ (use the quadratic formula if you need to).

The golden ration has now slipped into our generating function. We can then write:

$$G(x) = \frac{x}{\left(x - \frac{1+\sqrt{5}}{2}\right)\left(x - \frac{1-\sqrt{5}}{2}\right)}$$

as a partial fraction. To simplify the notation, let $\phi_+ = \frac{1+\sqrt{5}}{2}$ and $\phi_- = \frac{1-\sqrt{5}}{2}$.

89. Show that

$$G(x) = \frac{1}{\phi_+ - \phi_-}\left(\frac{1}{1-\phi_+ x} - \frac{1}{1-\phi_- x}\right)$$

90. Use problem 89 and the formula for the geometric series to show that

This is known as the Euler-Binet formula.

$$F_n = \frac{1}{\sqrt{5}}\left(\phi_+^n + \phi_-^n\right) = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

91. Program your graphing calculator or CAS to display the Fibonacci sequence.

92. Show that

$$\frac{F_{n+1}}{F_n}$$

approaches the golden ratio

Your CAS may make the calculations less tedious.

$$\frac{1+\sqrt{5}}{2}$$

For any three-term recurrence, an explicit formula can be written using this method.

93. (a) Show that $A_{n+1} = 2A_n + A_{n-1}$ has a generating function

$$\frac{-x}{x^2+2x-1}$$

(b) Following steps similar to those in the Fibonacci recurrence to get an explicit formula for A_n if $A_0 = 0$ and $A_1 = 1$.

94. (a) Show that $A_{n+1} = 3A_n + A_{n-1}$ has a generating function

$$\frac{-x}{x^2+3x-1}$$

(b) Following steps similar to those in the Fibonacci recurrence to get an explicit formula for A_n if $A_0 = 0$ and $A_1 = 1$.

95. Find the generating function and an explicit formula for the recurrence

$$A_{n+1} = 2A_n + 3A_{n-1}$$

96. Find the generating function and an explicit formula for the recurrence

$$A_{n+1} = \pi A_n + A_{n-1}$$

97. Find the generating function and an explicit formula for the recurrence

$$A_{n+1} = 2\pi A_n - A_{n-1}$$

98. Find the Fibonacci sequence in Pascal's triangle. Explain.

99. Calculate and conjecture

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n$$

for $n = 1 \ldots 4$

100. Calculate and conjecture

(a)

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(b)

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

(c)

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 8 \end{pmatrix}$$

Problems for Day 5

101. Recall the sequence of Chebyshev polynomials defined by

$$t_n(x) := \begin{cases} 1 & \text{if } n = 0 \\ x & \text{if } n = 1 \\ 2x t_{n-1}(x) - t_{n-2}(x) & \text{if } n > 1 \end{cases}$$

We have:

n	$t_n(x)$
0	1
1	x
2	$-1 + 2\,x^2$
3	$-3\,x + 4\,x^3$
4	$1 - 8\,x^2 + 8\,x^4$
5	$5\,x - 20\,x^3 + 16\,x^5$
6	$-1 + 18\,x^2 - 48\,x^4 + 32\,x^6$
7	$-7\,x + 56\,x^3 - 112\,x^5 + 64\,x^7$
8	$1 - 32\,x^2 + 160\,x^4 - 256\,x^6 + 128\,x^8$
9	$9\,x - 120\,x^3 + 432\,x^5 - 576\,x^7 + 256\,x^9$
10	$-1 + 50\,x^2 - 400\,x^4 + 1120\,x^6 - 1280\,x^8 + 512\,x^{10}$

Make them into *coefficients* of a power series in z: Think of x as a constant.

$$F(z) = 1 + xz + (-1+2x^2)z^2 + (-3\,x + 4\,x^3)z^3 + (1 - 8\,x^2 + 8\,x^4)z^4 + \ldots = \sum_{n=0}^{\infty} t_n(x)\,z^n$$

So this is a power series in z whose coefficients are polynomials in x. Show that

$$2xF(z) - zF(z) = 2x + \frac{F(z) - 1 - xz}{z}$$

102. So, in the notation of problem 101,

$$F(z) = \frac{1 - xz}{z^2 - 2xz + 1}$$

103. Keeping the same notation, show that

$$F(z) = \frac{A}{x - \alpha} + \frac{B}{x - \beta}$$

where

$$\begin{aligned} A &= \frac{-\left(-1 + x^2 + x\sqrt{-1 + x^2}\right)}{2\sqrt{-1 + x^2}} \\ B &= \frac{-\left(1 - x^2 + x\sqrt{-1 + x^2}\right)}{2\sqrt{-1 + x^2}} \\ \alpha &= x + \sqrt{x^2 - 1} \\ \beta &= x + \sqrt{x^2 - 1} \end{aligned}$$

104. Continuing, show that

$$F(z) = -\frac{A}{\alpha}\sum_{n=0}^{\infty}\frac{1}{\alpha^n}z^n + -\frac{B}{\beta}\sum_{n=0}^{\infty}\frac{1}{\beta^n}z^n$$

so that

$$t_n(x) = -\frac{A}{\alpha}\frac{1}{\alpha^n} + -\frac{B}{\beta}\frac{1}{\beta^n} = A_n(x) = \frac{1}{2}(x+\sqrt{x^2-1})^n + \frac{1}{2}(x-\sqrt{x^2-1})^n$$