9 *Do the Twist*

Alice and Bob have been checking their formulas for the Chebyshev polynomials to make sure that their recursion is correct. They have found that it works twice, so they know it always works.

1. If you have not done so already, do problems 14 to 17 from yesterday.

2. (a) What does the following recursion produce:

$$\begin{aligned} f(0) &= 0 \\ f(1) &= 1 \\ f(n) &= f(n-1) + f(n-2) \end{aligned}$$

(b) Write a recursion in a similar format for the Chebyshev polynomials, where the zeroth Chebyshev polynomials is $t_0(x)$ and the first is $t_1(x)$.

3. (a) What are the zeroes for the first few Chebyshev polynomials? You may use the graphs or what you know about the unit circle. Can you find a general formula for the zeroes of the nth Chebyshev polynomial?

(b) Based on your work from number 15 of problem set 8, what types of symmetry do you see in the graphs of each function? Explain why the symmetry exists.

Alice and Bob's teacher telephoned Pafnuty's father, Lev, to get an accurate spelling of family's last name for her records. Lev was not available as he was helping to fight off a French invasion.

4. (a) What happens to the signs of the coefficients in a given Chebyshev polynomial?
(b) Write a general formula for the powers of "x" in each term of the nth Chebyshev polynomial.

5. The nth Chebyshev polynomial is written as $t_n(x)$. If we want to talk about the kth coefficient of the nth Chebyshev polynomial, we can write it as $t_{n,k}$. For example, the $t_{4,2}$ is -8. Write the coefficients of the Chebyshev polynomials in the following table (you may add more rows and/or columns if you need to):

n	$t_{n,0}$	$t_{n,1}$	$t_{n,2}$	$t_{n,3}$	$t_{n,4}$	$t_{n,5}$	$t_{n,6}$	$t_{n,7}$	$t_{n,8}$	$t_{n,9}$	$t_{n,10}$
1											
2											
3											
4											
5											
6											
7											
8											
9											
10											

6. Can you find a function that can produce the numbers in the first column? The second column? The third column? Are your functions related to each other? Can you write the functions in terms of n and k?

The answer to all these questions is "Sure."

7. Describe patterns you see in the table.

8. Experiment with this function:

$$g(n) = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

Tabulate it between 1 and 20, for example.

Hint: If $\alpha = \frac{1+\sqrt{5}}{2}$, $\alpha^2 = \alpha + 1$.

9. Suppose that:

- $z = 2 + i$,
- $w = \frac{-1+i\sqrt{3}}{2}$

Plot each of the following on the complex plane. Use separate graphs for each.

(a) 1, ω, and ω^2 (b) 1, ω, and $\overline{\omega}$ (c) 1, ω, and $1+\omega$ (d) z and iz
(e) z and ωz (f) $1+\omega+\omega^2$ (g) $\omega+\omega^2$

10. Suppose f is a function on the complex numbers that adds $3-2i$ to everything:

$$f(w) = w + (3 - 2i)$$

Find:
(a) $f(1+5i)$ (b) $f(-4+6i)$ (c) $f(1)$ (d) $f(i)$

11. Consider (again) the function $f(w) = w+z$ with $z = 3-2i$.
(a) Plot w and $f(w)$ for 3 values of w.
(b) Describe how to get $f(w)$ from w, geometrically.

You choose the values of w. Not the ones used in problem 10!

12. Show that if z and w are complex numbers, the distance between z and w on the complex plane is $|z-w|$.

13. Let $\omega = \dfrac{-1+i\sqrt{3}}{2}$. Show that the triangle whose vertices are 1, ω, and ω^2 is equilateral.

14. Plot the set of all complex numbers that satisfy each condition (one graph for each problem):
(a) $|z| = 3$ (b) $|z| = 1$ (c) $|z| < 1$ (d) $|z| > 1$
(e) $|z| = \left|\frac{1}{z}\right|$ (f) $z^2 = z$ (g) $\overline{z} = z$ (h) $|z^2| = |z|$

15. Write each complex number as a product of a positive real number and a complex number of length 1:
(a) $4+3i$ (b) $5+12i$ (c) $1+i$
(d) $1-i$ (e) $2+3i$ (f) $-1+i\sqrt{3}$

Note: A complex number of length 1 is just a complex number whose absolute value is 1. (So, the length of its vector is 1.)

16. Show that any non-zero complex number can be written as the product of a positive real number and a complex number of length 1.

17. Use the complex numbers $z = 2+2i$ and $w = -1-2i$ to do the following.
(a) Plot z and w
(b) Find $z+w$ and $|z+w|$
(c) Plot $z+w$
(d) Find $|z|$, $|w|$, $|z|+|w|$
(e) Which is bigger, $|z|+|w|$ or $|z+w|$?

18. Use the picture below to explain the following theorem:

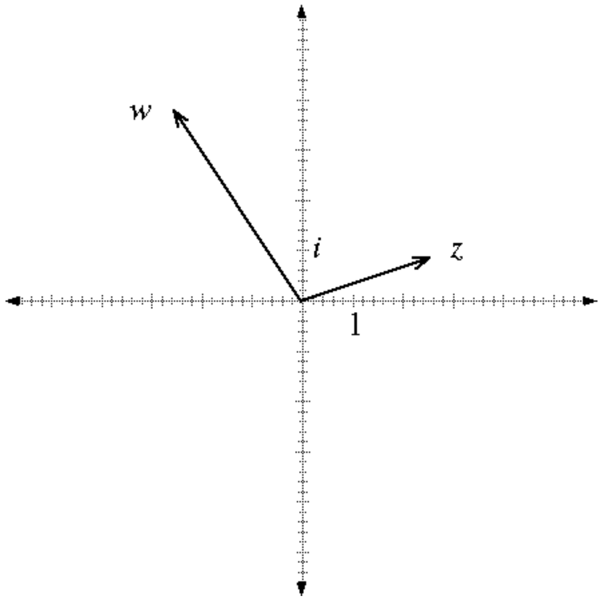
This theorem is sometimes called the "triangle inequality."

Theorem 1

If z and w are complex numbers,

$$|z+w| \leq |z| + |w|$$

Use geometry, not algebra, to explain what the theorem says.

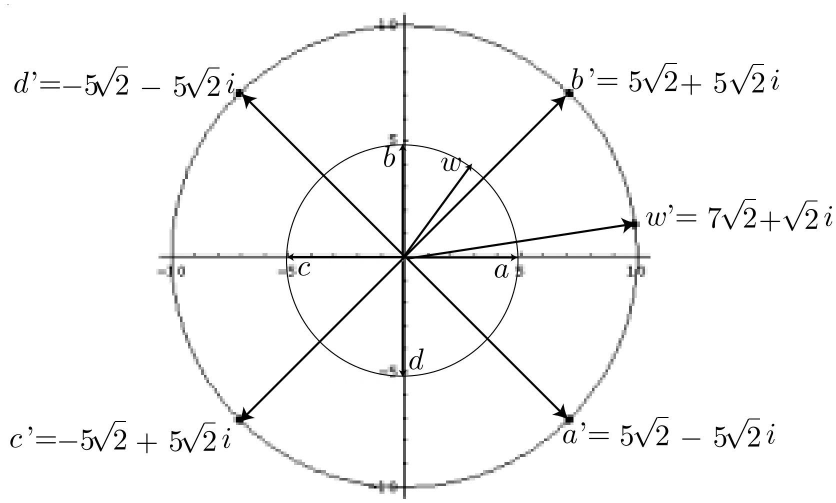


19. (a) Find 3 pairs of complex numbers such that $|z+w| = |z| + |w|$.

(b) Generalize to describe all pairs of complex numbers in which:

$$|z+w| = |z| + |w|$$

20. Let $\zeta = \dfrac{1+i\sqrt{3}}{2}$. Show that the powers of ζ lie on the vertices of a regular polygon in the complex plane.



Refer to the above example for these problems.

You'll get much more out of the above figure if you create it yourself. Start with the five points on the small circle, multiply each by $\sqrt{2}-\sqrt{2}i$, and plot the points you get.

21. (a) Why does multiplying by $\sqrt{2}-\sqrt{2}i$ move the points to the circle of radius 10, too?

(b) If you multiplied every point on the circle of radius 5 by $\sqrt{2}-\sqrt{2}i$, would the results lie on the circle of radius 10? Why or why not?

(c) Find four more complex numbers that would also move the points to circle of radius 10, under multiplication.

22. As you would expect, multiplying by 2 is not the same as multiplying by $\sqrt{2}-\sqrt{2}i$. What differences do you notice?

23. (a) Choose five points on the circle of radius 1 and graph them.

(b) Multiply the points by 2 and graph the results. Describe how multiplying by 2 affects the location of the points.

(c) Multiply the points by $\sqrt{2}-\sqrt{2}i$ and graph the results. Describe how multiplying by $\sqrt{2}-\sqrt{2}i$ affects the location of the points.

24. Choose five points on the circle of radius 3, and repeat steps (b) and (c) of problem 23. Make a general conjecture about how multiplying by 2 and multiplying by $\sqrt{2}-\sqrt{2}i$ affect the locations of points in the complex plane.

25. Describe (as completely as you can) how multiplying by z affects the locations of points in the complex plane.

(a) $z = \frac{1}{4}$ (b) $z = -1$
(c) $z = 3$ (d) $z = -5$
(e) $z = i$ (f) $z = -i$
(g) $z = -3i$ (h) $z = 1 + i$
(i) $z = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ (j) $z = \sqrt{3} - i$

You may want to choose some points on the circle of radius 1, as you did in problem 23, to form conjectures.

26. What complex number (in $a + bi$ form) do you get if you rotate $2 + 3i$ through an angle of

(a) 90° (b) 60° (c) 180° (d) 215°
(e) 150° (f) −45° (g) 390°

27. Find two "nice" complex numbers whose product is

(a) i (b) $\cos 75° + i \sin 75°$ (c) -1
(d) $\cos 165° + i \sin 165°$ (e) $\cos 105° + i \sin 105°$ (f) $\frac{1}{2} + \frac{\sqrt{3}}{2}i$

28. Plot the first 10 powers of each complex number, starting with $z^0 = 1$ (the zeroth power)

(a) $z = i$ (b) $z = 2i$
(c) $z = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ (d) $z = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$
(e) $z = 1 + i$ (f) $z = \frac{3}{10} - \frac{3}{10}i$
(g) $z = -\frac{3}{5} - \frac{4}{5}i$ (h) $z = 3 + 4i$
(i) $z = \cos 20° + i \sin 20°$ (j) $z = -\frac{1}{4} - \frac{1}{3}i$
(k) $z = \cos 72° + i \sin 72°$
(l) $z = \cos(-36°) + i \sin(-36°)$

29. Describe the picture you get when you plot the powers of a complex number z $(1, z, z^2, z^3, z^4, \ldots)$ if:

(a) $|z| = 1$ (b) $|z| > 1$ (c) $|z| < 1$

30. For $z = \cos 72° + i \sin 72°$, find

(a) $\arg(z^2)$ (b) $\arg(z^3)$ (c) $\arg(z^5)$ (d) $\arg(z^{10})$ (e) $\arg(z^n)$

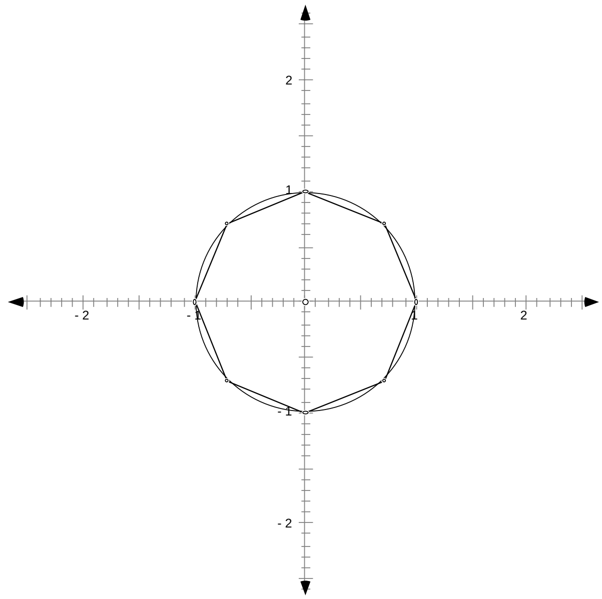
31. The powers of $\cos 30° + i \sin 30°$ lie on a regular polygon in the complex plane. How many sides does this polygon have?

32. Find a complex number whose powers (starting with the zeroth power) form the vertices of a regular polygon with n sides.

(a) $n = 3$ (b) $n = 5$ (c) $n = 7$ (d) $n = 4$ (e) $n = 6$

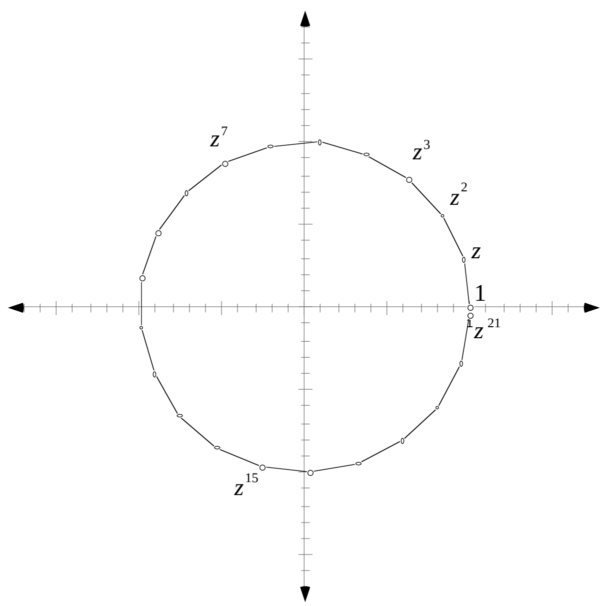
33. Here's a picture of a regular octagon inscribed in the unit circle on the complex plane. The eight vertices are complex numbers. One of these vertices is $1 + 0i$. Find the others.

Can you write your answers without using sines and cosines?



Show that all the vertices are powers of one vertex.

34. The graph below shows the plot of the powers (from 0 to 21) of
$z = \cos 17° + i \sin 17°$.



z^{21} doesn't make it. z^{22} would be back in the first quadrant.

(a) Why aren't these powers of z vertices of a regular polygon?

(b) Will the powers of z ever come back to 1? If so, how many times around will it take? If not, why?

(c) Describe all θs for which the powers of $z = \cos\theta + i\sin\theta$ are vertices of a regular polygon.

35. Show that if $z = \cos 72° + i \sin 72°$:

(a) $\zeta^4 = \overline{\zeta} = \dfrac{1}{\zeta}$ (b) $\zeta^3 = \overline{\zeta^2} = \dfrac{1}{\zeta^2}$ (c) $\zeta + \zeta^4 = 2\cos 72°$

36. Consider our equation $x^5 - 1 = 0$

(a) Factor $x^5 - 1$.

(b) Show $\zeta^4 + \zeta^3 + \zeta^2 + \zeta = -1$.

37. Let $a = \zeta + \zeta^4$ and $b = \zeta^2 + \zeta^3$.

(a) Show $a + b = -1$. (b) Show that $ab = -1$.

38. Problem 37 says that a and b are two numbers whose sum is -1 and whose product is -1.

(a) Show that they are therefore roots of the equation

$$x^2 + x - 1 = 0$$

(b) Use this equation to find a and b.

39. (a) Show that:

$$\cos 72° = \frac{-1 + \sqrt{5}}{4}$$

(b) What is $\sin 72°$ (exactly)?

40. Write algebraic expressions for each of the roots of $x^5 - 1 = 0$ that do not involve cosine and sine.