10 Loaded Dice

Alice and Bob, each armed with a CAS, set out to do battle against the loaded dice. I suggest that you take the same precautions.

1. What is the distribution of sums on two regular six-sided dice? Three dice? Four dice? What is the most common sum(s)? What is the probability of rolling the most common sum? Explore this using your CAS (TI-89/92) and the following polynomial, $(x+x^2+x^3+x^4+x^5+x^6)^n$. If you need to, verify by any means you know.

2. If one die is fair and the other is weighted (1,1,3,4,5,6), what is the new distribution?

3. If two dice are fair and another is weighted (1,1,3,4,5,6), what is the new distribution? What sums do we not get here that we got with 3 fair dice.

4. What is the distribution on three dice labelled: (0,2,3,4,5,5), (0,1,1,2,2,2), and (1,2,3,6,6,6)?

5. In general, what do the coefficients in the polynomials represent? What do the exponents in the polynomials represent?

The polynomial you used to get the answers to #1, $(x+x^2+x^3+x^4+x^5+x^6)^n$, is also known as the generating function for the distribution of sums on regular six-sided dice. The other functions you used in numbers 2, 3 and 4 are also generating functions.

Alice and Bob take a break from the dice and ask their teacher if there are any other activities that involve exponents. Their teacher gives them the following.

6. For each of the following infinite sums, write it's fractional equivalent:
   (a) $2*(1/100)+2*(1/100)^2+2*(1/100)^3+\ldots$
   (b) $1*(1/100)+1*(1/100)^2+2*(1/100)^3+3*(1/100)^4\ldots$
   (c) $3*(1/10)+6*(1/10)^2+9*(1/10)^3+\ldots$
   (d) $2*(1/11)+8*(1/11)^2+32*(1/11)^3+\ldots$

#6(b). The coefficients are the Fibonacci numbers.

7. (a) Show that $1+x^1+x^2+x^3+x^4+\ldots=\frac{1}{1-x}$. Is this true for all x?
   (b) Express $1+(1/5)+(1/5)^2+(1/5)^3+(1/5)^4+\ldots$ as a fraction.
   (c) Express $1+(2/7)+(2/7)^2+(2/7)^3+(2/7)^4+\ldots$ as a fraction.
   (d) Did you see that part (b) is equal to $\frac{1}{1-(1/5)}$.
   (e) Express $1+(5/4)+(5/4)^2+(5/4)^3+(5/4)^4+\ldots$ as a fraction.

8. In number 7, $\frac{1}{1-x}$ is known as the generating function for a geometric series. Dividing 1 by (1-x) gives us back the $1+x^1+x^2+x^3+x^4+\ldots$ that we started with. Do the long division to show that the above is true. Are there any restrictions on the generating function $\frac{1}{1-x}$?

Alice and Bob are not too big on long division, they like the adding better. In fact, they are inspired by the wonders of adding exponents and can't resist doing more.

9. What is the distribution of the different combinations (order does not matter) you get when you flip two coins? Three coins? Four coins? Are you seeing a pattern here? How about when you flip *n* coins?

#9. "Heads, Heads, Tails" is the same as "Tails, Heads, Heads."

10. Create a polynomial, which, when raised to different powers, will produce the distribution you obtained in number 9 (use "h" for heads and "t" for tails). This polynomial is a generating function for the distribution you get when you flip two coins.

11. How many combinations of nickels and dimes can you use to make 25 cents? Do this problem any way you feel comfortable.

12. (a) Explain how you get the x^{25} coefficient in the following product: $(1+x^5+x^{10}+x^{15}+x^{20}+x^{25})(1+x^{10}+x^{20})$
(b) What is the significance of the 1's in the above polynomial?

13. Use a generating function to show how many different ways can you make change for 25 cents using nickels, dimes and quarters.

14. Use the generating function idea to show how many ways there are to make 50 cents using nickels, dimes and quarters?

#14. You do not actually need to find the answer.

15. Explain how you would go about finding out how many ways to make change for a dollar using pennies, nickels, dimes and quarters (half dollars too, if you must).

16. (a) Find the generating function for $2x+4x^2+8x^3+16x^4+32x^5+\ldots$
(b)Write the following as a fraction: $2(1/100)+4(1/100)^2+8(1/100)^3+16(1/100)^4+32(1/100)^5+\ldots$
(c)Write the following as a fraction: $2(1/16)+4(1/16)^2+8(1/16)^3+16(1/16)^4+32(1/16)^5+\ldots$
(d)Find the generating function for $8x+16x^2+32x^3+64x^4+128x^5+\ldots$

17. Write a recursive formula for each of 16(a),(b),(c) and (d). How are the formulas related?

Pafnuty was not partial to all this business of adding exponents. He was partial toward both adding and subtracting.

18. (a) Write 1/6 as the difference of two fractions whose denominators are less than six. Try it for 2/15, 1/15, and 7/15 (the denominators need to be less than 15).

(b) Write $\frac{2x-3}{x^2-3x-10}$ as the sum or difference of two fractions with denominators of degree 1.

19. (a) Write each of 16(a),(b),(c) and (d) in summation notation.
(b) Do the same for each of 7(a)-(e).

These problems on loaded dice are extra problems.

20. What distribution of sums do you get when you roll 2 dice, each labeled (2, 3, 3, 4, 4, 5)? Look familiar? Can you label another set of dice that follow a similar pattern (binomial coefficients)?

21. On 2 six-sided dice, labeled with any combination of 0-6, how would the dice have to be labeled so that the sums 1-12 are equally likely to come appear?

George ordered Alice and Bob to label a pair of six-sided dice whose distribution of sums would be the same as regular six-sided dice.

22. How would those dice be labeled? You may use any natural numbers you like to label the sides. Repeats and numbers greater than six are allowed. The dice don't have to be labeled the same.

23. What is/are the most common sum(s) on *n* regular six-sided dice are rolled? What is the frequency of the sum(s)? What is the probability of the most frequently occuring sum(s)?

24. I know I said not to do it earlier. But, how many ways are there to make change for a dollar?

25. Factor (in at least two ways) the expansion of $(x+x^2+x^3+x^4+x^5+x^6)^n$. How does this relate (or help you solve) number 22.

26. Explore what happens when you label the face of dice with negative numbers. Do the same rules hold here that held for positive numbers? Can you label another set of dice

(using the integers) which, when rolled, will give the same distribution as regular dice?