11 Adding and Subtracting Infinite Series

1. One die labeled (1,3,4,5, (1,2,2,3,3,4) are thrown. What is the distribution of the sums on the two dice? What if two of **each** dice are thrown? Look familiar?

#1. If you desire further exploration, look back at #25 from yesterday, but not right now.

2. Use polynomials to show how many ways are there to make change for a dollar using nickels, dimes and quarters. Look back at number 11 and 12 from yesterday if you need to.

3. Explain how you'd get the coefficients of x^{34} and x^{14} in the product: $(1+x^5+x^{10}+x^{15}+x^{20}+x^{25}+\square\ +x^{50})(1+x^8+x^{16}+x^{24}+x^{32}+x^{40})$.

4. The post office only has 5 and 8 cent stamps this morning. Which amounts less than 50 cents can you make in at least one way (it might take less time to write which ones you can't make)? What is the largest amount you cannot make with 5 and 8 cent stamps?

#4. Attempt this using polynomials similar to those in number 3.

5. The post office only has 6 and 11 cent stamps today. What is the largest amount (it is less than 100 cents) you cannot make with 6 and 11 cent stamps?

6. Write the generating function for each of the following:
(a) $2 + 4x + 8x^2 + 16x^3 + 32x^4 + 64x^5 + 128x^6 + \square$
(b) $4x + 8x^2 + 16x^3 + 32x^4 + 64x^5 + 128x^6 + \square$
(c) $8x + 16x^2 + 32x^3 + 64x^4 + 128x^5 + 256x^6 + \square$
(d) $3x + 9x^2 + 27x^3 + 81x^4 + 243x^5 + 729x^6 + \square$
(e) $27 + 81x + 243x^2 + 729x^3 + 2187x^4 + 6561x^5 + \square$
Write down the recursion for the n^{th} term of each sequence. Write down your observations about the generating functions and the recursion for these 5 examples. What changes? What stays the same? Write down a function, for each of the above, that will generate its n^{th} coefficient.

#6. The generating function for: $1+x^1+x^2+x^3+x^4+\ldots$ is $\frac{1}{1-x}$.

7. Using the generating functions you obtained in number 6 to express the following as fractions.
(a) $2 + 4(1/3) + 8(1/3)^2 + 16(1/3)^3 + 32(1/3)^4 + 64(1/3)^5 + \square$
(b) $4(2/10) + 8(2/10)^2 + 16(2/10)^3 + 32(2/10)^4 + 64(2/10)^5 + \square$
(c) $8(1/8) + 16(1/8)^2 + 32(1/8)^3 + 64(1/8)^4 + 128(1/8)^5 + \square$
(d) $3(1/5) + 9(1/5)^2 + 27(1/5)^3 + 81(1/5)^4 + 243(1/5)^5 + \square$
(e) $27 + 81(1/1000) + 243(1/1000)^2 + 729(1/1000)^3 + \square$

and if you are still curious, try:

(f) $2+4(17/36)+8(17/36)^2+16(17/36)^3+32(17/36)^4+64(17/36)^5+\cdots$

Alice, Bob, and Pafnuty teamed up to do long division. They quickly grew tired of long division as the remainders seemed to go on forever.

8. Practice your long division on the following (4 terms of the remainder should suffice):

(a) $\frac{1}{1-3x}$ (b) $\frac{2}{1-3x}$ (c) $\frac{1}{1-5y}$ (d) $\frac{1}{1-(1/2)x}$

*(e) $\frac{1}{1-ax}$ *(f) $\frac{b}{1-ax}$

Pafnuty explained to Alice and Bob that numbers 8(e) and 8(f) are the general forms for the generating functions of an infinite geometric series.

9. Using the general forms write the infinite series (at least the first 4 terms) that is represented by each of the following generating functions (do not use long division to do this):

(a) $\frac{1}{1-5x}$ (b) $\frac{1}{1-(2/3)x}$ (c) $\frac{1}{1-(1/7)x}$

(d) $\frac{2x}{1-5x}$ (e) $\frac{2a}{1-ax}$

For each of the above, write a closed formula that will produce the n^{th} coefficient in the infinite series.

10. Use the general forms to more quickly create generating functions for the following:

(a) $1 + 6x + 36x^2 + 216x^{43} + 1296x^4 + 7776x^5 + \cdots$

(b) $1 + (1/2)x + (1/4)x^2 + (1/8)x^3 + (1/16)x^4 + \cdots$

(c) $2 + 2(1/2)x + 2(1/4)x^2 + 2(1/8)x^3 + 2(1/16)x^4 + \cdots$

(d) $1 + (1/3)x + (1/9)x^2 + (1/27)x^3 + (1/81)x^4 + \cdots$

(e) $1 + (2/3)x + (4/9)x^2 + (8/27)x^3 + (16/81)x^4 + \cdots$

(f) $6x + 36x^2 + 216x^3 + 1296x^4 + 7776x^{65} + \cdots$

(g) $6x^2 + 36x^3 + 216x^4 + 1296x^5 + 7776x^6 + \cdots$

11. Given Fibonacci numbers (0,1,1,2,3,5, …) and Lucas numbers (2,1,3,4,7,11, …). What do you get when you subtract the nth Finonacci number from the nth Lucas number?

#11. Lucas numbers are formed by the same recursion as Fibonacci numbers, but start with 2 and 1.

PARTIAL FRACTIONS

In general, $\frac{a}{xy}$ can be written as $\frac{A}{x} + \frac{B}{y}$. Solve a system of equations for A and B to get the answer(s).

12. Here are a few to try.

(a) Write 1/14 as the difference of two fractions whose denominators are less than 14. Try it for 5/14 and 9/14 (the denominators need to be less than 14).

(b) Write $\frac{6-13x}{15x^2-8x+1}$ as the sum or difference of two fractions with denominators of degree 1.

(c) Write $\frac{7x+45}{x^2+5x-24}$ as the sum or difference of two fractions with denominators of degree 1.

*Two more examples are in #18 from yesterday.

13. From your work in number 6(a), you know the generating function for $2+4x+8x^2+16x^3+32x^4+64x^5+128x^6+\ldots$
Use your knowledge of the generating function to show what you need to substitute for x to make the sum 5/3. What do you substitute for x to make the sum 5/2? How about to make the sum 10? 20? 40? 5/4? 5/5? 5/6?

These are problems on stamps and loaded dice are extra.

14. What amounts can you make if you only had 6, 5 cent stamps and only 10, 8 cent stamps?

15. Using only 5 and 8 cent stamps. How many ways are the to make 40 cents? 52 cents? 60 cents? Are you able to do it with generating functions?

16. Today, the post office has stamps with values “a” and “b.” The stamps do not share any factors other than 1 (a and b are relatively prime). What is the biggest value you can’t make?

17. What distribution of sums do you get when you roll 2 dice, each labeled (2, 3, 3, 4, 4, 5)? Look familiar? Can you label another set of dice that follow a similar pattern (binomial coefficients)?

18. On 2 six-sided dice, labeled with any combination of 0-6, how would the dice have to be labeled so that the sums 1-12 are equally likely to come appear?

George ordered Alice and Bob to label a pair of six-sided dice whose distribution of sums would be the same as regular six-sided dice.

19. How would those dice be labeled? You may use any natural numbers you like to label the sides. Repeats and numbers greater than six are allowed. The dice don't have to be labeled the same.