14 And now we've come full triangle.

1. Produce the power series for the generating function $\frac{1}{1-x}$ (you may use "talyor" or long division). Let's call the coefficient of x^0, f(0). Let's call the coefficient of x^1, f(1). Let's call the coefficient of x^2, f(2). So, the coefficient of x^n can be called f(n). For this example all the coefficients are 1. Find a polynomial of lowest degree that agrees with this function.

2. Produce the power series for the generating function $\frac{x}{(1-x)^2}$ (you may use "talyor" or long division). Let's call the coefficient of x^0, f(0). Let's call the coefficient of x^1, f(1). Let's call the coefficient of x^2, f(2). So, the coefficient of x^n can be called f(n). Build a table and use the "delta" to find a polynomial of lowest degree that agrees with this function.

3. Produce the power series for the generating function $\frac{x^2}{(1-x)^3}$ (you may use "talyor" or long division). Using the same method from problem 1 and 2 to find a polynomial of lowest degree that agrees with this function.

4. Produce the power series for the generating function $\frac{x^3}{(1-x)^4}$ (you may use "talyor" or long division). Using the same method from problem 1 and 2 to find a polynomial of lowest degree that agrees with this function.

5. Have you seen these polynomials before? If so, do you remember what they are called?

6. What polynomial will produce the coefficients produced by the generating function $\frac{x^n}{(1-x)^{n+1}}$?

7. What recursion will produce the coefficients of power series produced by the following generating functions (each of the generating functions below has its own recursion):

$A(\mathrm{x}) = \frac{3-x}{1-2x-x^2}$

And now we've come full triangle

Here is the power series it produces.

$3+5x+13x^2+31x^3+75x^4+\cdots$

$$B(\mathrm{x}) = \frac{6-x}{1-2x+x^2}$$

Here is the power series it produces.

$6+11x+16x^2+21x^3+26x^4+\cdots$

$$C(\mathrm{x}) = \frac{x}{1+2x-29x^2+42x^3}$$

Here is the power series it produces.

$0+1x-2x^2+33x^3-166x^4+1373x^5+\cdots$

8. Did the pattern for finding the recursion of the power series produced by a generating function with a quadratic denominator hold for generating function with a cubic denominator?

9. Find a recursion that will produce the coefficients of the power series produced by $\frac{x}{(1-x)^2}$. Do the same for $\frac{x^2}{(1-x)^3}$. Then again for $\frac{x^3}{(1-x)^4}$. What recursion will produce the coefficients of the power series produced by $\frac{x^n}{(1-x)^{n+1}}$.

If you are up for some rewarding algebra go to number 10 (or skip it if you prefer and go to problem 11):

10.

Chebyshev Polynomials

$C_0 = 1$

$C_1 = t$

$C_2 = 2t^2-1$

$C_3 = 4t^3-3t$

$C_4 = 8t^4-8t^2+1$

Use **The Root Theorem** to find the function that generates the n^{th} Chebyshev polynomial. Find the denominator of the generating function (or the whole generating function if you are glutton for punishment).

And now we've come full triangle

$$\begin{cases} C_0 = 1 \\ C_1 = t \\ C_n = 2tC_{n-1} - C_{n-2} \end{cases}$$

$$1 + tx + (2t^2 - 1)x^2 + (4t^3 - 3t)x^3 + (8t^4 - 8t^2 + 1)x^4 + \cdots$$

11. The "closed" formula for the n^{th} Chebyshev polynomial can be written as $C_n(t) = \frac{1}{2}\left(t + \sqrt{t^2 - 1}\right)^n + \frac{1}{2}\left(t - \sqrt{t^2 - 1}\right)^n$.

The "closed" formula for the n^{th} Fibonacci number can be written as $F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$.

Show that when t=$\frac{i}{2}$, $C_n(t) = \frac{i^n}{2}(F_n + 2F_{n-1})$ is true for n=1,2,3.
Why is this the case?

13. Experiment with the following matrices. Notice anything interesting?

(a) $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} * \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} * \begin{pmatrix} 8 \\ 5 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n$

14. Using matrices, can you say anything or prove anything about the n^{th} Fibonacci number?

15. Make a matrix or matrices that produce the Lucas numbers.

16. Experiment with the following matrices (determinants are the way to go here):

(a) $\begin{bmatrix} x & 1 \\ 1 & 2x \end{bmatrix}$ (b) $\begin{bmatrix} x & 1 & 0 \\ 1 & 2x & 1 \\ 0 & 1 & 2x \end{bmatrix}$ (c) $\begin{bmatrix} x & 1 & 0 & 0 \\ 1 & 2x & 1 & 0 \\ 0 & 1 & 2x & 1 \\ 0 & 0 & 1 & 2x \end{bmatrix}$

17. Find a set of matrices that will produce the solutions to yesterday's stamp problem.

18. Can you find a set of matrices that will produce the results of any two-term recursion.