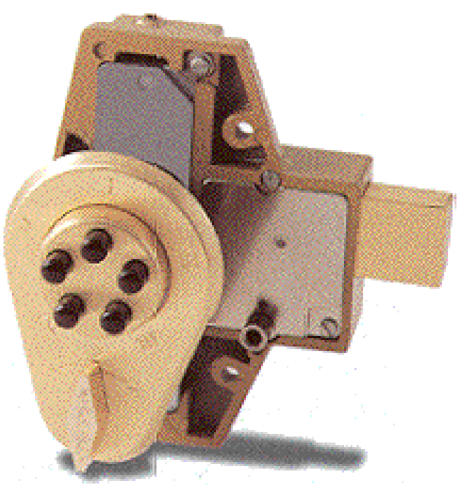
The Simplex Lock

Here's a problem that will weave its way in and out of many of the topics we discuss this summer. We'll start working on it right now, but don't feel that you have to get the "right answer" right away. We'll keep coming back to it, even after you might be satisfied that you know what's going on. There's a lot to discover in this problem.

The Simplex company makes a combination lock that is used in many public buildings. It comes in several versions. Here are two:

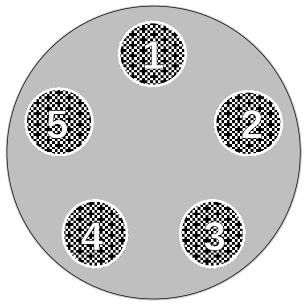


These 5-button devices are purely mechanical (no electronics). You can set the combination using the following rules:

1. A combination is a sequence of 0 or more pushes, each push involving at least one button.
2. Each button may be used at most once (once you press it, it stays in).

There's one possible no-push combination: the door's just already unlocked. Not a great combination, but it counts.

3. Each push may include any of the buttons that haven't been pushed yet, up to and including all remaining buttons.
4. The combination does *not* need to include all buttons.
5. When two or more buttons are pushed at the same time, order doesn't matter.



Artist's rendition of a Simplex Lock

Here are some possible combinations:

Notation: $\{\{1,2\},\{3\}\}$ means "press 1 and 2 together, then press 3."

- $\{\{1,3\},\{4\}\}$
- $\{\{1,2\},\{3\}\}$
- $\{\{2\},\{1\},\{3\}\}$
- $\{\{1,2,4\},\{3,5\}\}$
- $\{\{1,2,3,4,5\}\}$
- $\{\{1,2\},\{4\},\{3,5\}\}$
- $\{\{3\},\{1,2\}\}$
- $\{\ \}$
- $\{2\}$

The company advertises thousands of combinations, and (as we say to our students), the question is, "Is the company telling the truth?"

PROBLEM

How many combinations are there on a 5-button Simplex lock?

1 *Trains*

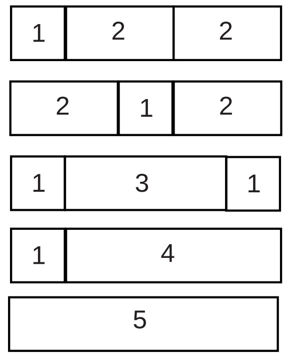
Hey, welcome to the class. We know you'll learn a lot of math here—maybe some new tricks, maybe some new perspectives on things you're already familiar with. A few things you should know about how the class is organized:

- Don't worry about answering all the questions, ever. If you're answering every question, we haven't written the problem sets correctly.
- Don't worry about getting to a certain problem number. Some participants have been known to spend the entire session working on one problem (and perhaps a few of its extensions or consequences).
- Have fun! Make sure you're spending time working on problems you're interested in. Feel free to skip problems that you're already sure of. Relax and enjoy!
- Each day starts with problems like the ones below, intended to be picked up on regardless of how much or little work you've done on prior sets. Try them as a starting point.

Okay, so let's get started. The first set of problems revolve around "trains" of rods. You can use rods of integer sizes to build "trains" that all share a common length.

Think of the rods as Cuisenaire Rods if you know what these are. Unless you are told otherwise, you have an unlimited supply of all the rod types.

A "train of length 5" is a row of rods whose combined length is 5. Here are some examples:



Notice that the 1-2-2 train and the 2-1-2 train contain the same rods but are listed separately. If you use identical rods in a different order, this is a separate train.

1. How many trains of length 5 are there?

2. Find a formula for the number of trains of length n. Come up with a convincing reason that your rule is correct.

(Assume you have rods of every possible integer length available.)

3. Create an algorithm that will generate all the trains of length n.

4. How many trains of length n are there that use *only* cars of length 1 and 2? Find a general rule, and explain why your rule works.

5. How many trains of length 11 are there that use only cars of length 1, 2, and 3?

Neat Stuff.

This section includes a variety of problems each day, which range from practice to extension. Pick and choose problems to work on, depending on your background and focus. Don't be surprised to see problems here repeated in later sets; that's our way of suggesting you check it out sometime.

6. Suppose there are three flavors of ice cream: pistachio, strawberry, and chocolate. How many different three-scoop cones can you make using each of these flavors exactly once?

Note that in a *cone*, it is important which scoop is on top. Thus, a pistachio-strawberry-chocolate cone is different from a strawberry-chocolate-pistachio cone.

7. Suppose you want a four-scoop cone with one scoop each of pistachio, strawberry, chocolate, and butter pecan. After your release from the mental hospital, how many different cones could you make with these flavors? Explain your reasoning carefully.

8. (a) How many different cones can you make from 5 scoops of different flavors?

(b) How many different cones can you make from n scoops of different flavors?

9. Cows (which is a must-visit in PC) serves 24 different flavors of ice cream.

Gosh, if only you'd get to go there sometime soon, like maybe later today.

(a) How many different three-scoop cones can you make at Cows?

(b) How many different four-scoop cones can you make at Cows?

(c) Find a rule for determining the number of different cones available at Cows in terms of the number of scoops on the cone.

10. In a *bowl* of ice cream, the order of the scoops does not matter. Therefore, a chocolate-vanilla bowl is the same as a vanilla-chocolate bowl.

If you've taught the IMP *Cones and Bowls* unit, you probably saw this one coming. . .

(a) At a certain ice cream shop, you can make 465 different two-scoop bowls of ice cream. How many different two-scoop cones can you make? Explain how you know.

(b) Find the number of flavors offered at this ice cream shop.

11. If you can make 220 different three-scoop bowls of ice cream, how many different three-scoop cones can you make?

12. (a) If you can make 210 different four-scoop bowls of ice cream, how many different four-scoop cones can you make?

(b) If you can make 3024 different four-scoop *cones* of ice cream, how many different four-scoop *bowls* can you make?

13. If you can make 55440 different five-scoop cones, how many different five-scoop bowls can you make?

14. (a) At Cows (where there are 24 flavors), how many different five-scoop bowls can be made?

(b) Find a rule for determining the number of different bowls of k scoops at Cows.

Tough Stuff.

This section has some difficult problems! Try these if you're up for a challenge or already feel pretty confident about the problems in the rest of the set. Our guarantee: something challenging every day.

15. If you made all the trains of length 5, how many cars of length 1 were used? length 2, 3, 4, 5? See if you can find a general rule for the number of cars of length k you'd need to make all the trains of length n.

16. What's the *average* (mean) length of car used when you make all the trains of length 5? Is there a general rule at work here? Can you justify it?

17. How many three-scoop bowls could you make at Cows if you *were* allowed to duplicate flavors? Is there a general rule at work?

2 *More Trains*

We thought of naming this set "Trains 2: Electric Boogaloo", but thought better of it.

1. How many trains of length 6 are there?
2. How many trains of length 6 are there that use exactly one rod? two rods? three rods?
3. Celeste shows you a full chart of all the trains of length 4. Try to figure out a way to use the chart of trains of length 4 to generate all the trains of length 5.

Hm, supposedly there are twice as many trains of length 5...

4. How many trains of length 10 are there that use *only* cars of length 1 and 2?
5. Using *only* rods of lengths 2 and 3, how many trains of length 11 can be made?
6. Make a table of how many trains of length n can be made using *only* rods of length 2 and 3, for n from 1 to 11. Is there a rule you could use to continue the table?

Neat Stuff.

These problems may look familiar, since they are the same as yesterday's!

Note that in a *cone*, it is important which scoop is on top. Thus, a pistachio-strawberry-chocolate cone is different from a strawberry-chocolate-pistachio cone.

7. Suppose there are three flavors of ice cream: pistachio, strawberry, and chocolate. How many different three-scoop cones can you make using each of these flavors exactly once?
8. Suppose you want a four-scoop cone with one scoop each of pistachio, strawberry, chocolate, and butter pecan. After

your release from the mental hospital, how many different cones could you make with these flavors? Explain your reasoning carefully.

9. (a) How many different cones can you make from 5 scoops of different flavors?

(b) How many different cones can you make from n scoops of different flavors?

10. Does problem 9 generally get easier or harder if you're allowed to repeat flavors within a cone? Why?

11. Cows (which, we said yesterday, is a must-visit in PC) serves 26 different flavors of ice cream.

(a) How many different three-scoop cones can you make at Cows, if you never use the same flavor twice?

(b) How many different four-scoop cones can you make at Cows, if you never use the same flavor twice?

(c) Describe a rule for determining the number of different cones available at Cows in terms of the number of scoops on the cone.

12. In a *bowl* of ice cream, the order of the scoops does not matter. Therefore, a chocolate-vanilla bowl is the same as a vanilla-chocolate bowl.

It's like the lock: pushing #4 and then #2 is like a cone; order matters. Pushing *both* #4 and #2 at once is like a bowl.

(a) At a certain ice cream shop, you can make 465 different two-scoop bowls of ice cream. How many different two-scoop cones can you make? Explain how you know.

(b) Find the number of flavors offered at this ice cream shop.

13. If you can make 220 different three-scoop bowls of ice cream, how many different three-scoop cones can you make?

14. (a) If you can make 210 different four-scoop bowls of ice cream, how many different four-scoop cones can you make?

(b) If you can make 3024 different four-scoop *cones* of ice cream, how many different four-scoop *bowls* can you make?

15. If you can make 55440 different five-scoop cones, how many different five-scoop bowls can you make?

16. (a) At Cows (where there are 26 flavors), how many different five-scoop bowls can be made?
(b) Find a rule for determining the number of different bowls of k scoops at Cows.

Please, a five-scoop bowl should only be ordered for mathematical purposes.

17. Does problem 16 generally get easier or harder if you're allowed to repeat flavors within a bowl? Why?

Tough Stuff.

18. If you made all the trains of length 5, how many cars of length 1 were used? length 2, 3, 4, 5? See if you can find a general rule for the number of cars of length k you'd need to make all the trains of length n.

19. What's the *average* (mean) length of car used when you make all the trains of length 5? Is there a general rule at work here? Can you justify it?

20. In a coin-flipping game, you flip a fair coin (heads or tails) ten times. If you flip heads twice in a row at any point during the game, you lose. Find the probability that you win at this game.

3 *Even More Trains*

Or, Trains 3: Judgment Day.

1. (a) Make a table for $n = 2$ to 8 for the number of trains of length n that use *exactly* two rods.
(b) Repeat for *exactly* three rods.

2. Complete this table, with the train length on the vertical and the number of rods on the horizontal:

 # Rods Used

	1	2	3	4	5	6	7
1	1						
2	1	1					
3	1		1				
4	1			1			
5	1				1		
6	1	5	10			1	
7	1						1

3. Physically make all the trains of length 6 that use exactly two rods, and (separately, without destroying the two-rod trains) all the trains of length 6 that use exactly three rods.

4. Can you think of a way to use the trains you built in problem 3 to make all the trains of length 7 that use exactly three rods?

5. You've got an unlimited supply of rods 2, 3, and 5. How many different trains of length 12 can you make with all these rods?

6. How many trains of length 10 can you make with *no* rods of length 1?

In other words, use only rods of length 2, 3, 4, or more.

Neat and/or Needed Stuff

7. At the world's largest ice cream store, you can order 3,464,840 bowls of ice cream with four different-flavored scoops. *Without* figuring out how many flavors there are, can you describe a way to find how many four-scoop cones of ice cream are available?

8. (a) Suppose you can make 156 different two-scoop cones at a certain ice cream shop. How many different flavors does this shop offer?

(b) Suppose you can make 2730 different three-scoop cones at a certain ice cream shop. How many different flavors does this shop offer?

9. Let n and r be non-negative integers with $n \geq r$. Define

$${}_nP_r = \frac{n!}{(n-r)!}$$

Explain why ${}_nP_r$ is the number of *permutations* (or, cones) of n things taken r at a time.

By the way, 0! is 1. We'll talk about some convincing reasons why it makes sense.

10. Using the definition above, find ${}_nP_0$. Explain using the ice cream analogy.

When I was 4 years old, this is what I ordered at the ice cream shop. I loved it!

11. What's a simpler rule for ${}_nP_n$?

12. Let n and r be non-negative integers with $n \geq r$. Define

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Explain why $\binom{n}{r}$ is the number of r-scoop bowls you can make with n scoops of ice cream.

Note: We sometimes write ${}_nC_r$ instead of $\binom{n}{r}$.

13. Using the definition, find $\binom{n}{0}$. Explain using the ice cream analogy.

14. What's a simpler rule for $\binom{n}{n}$? Explain using ice cream. Mmmm, fattening.

15. A pizza store offers 10 kinds of toppings.

(a) Suppose you want exactly 2 toppings on your pizza. How many different pizzas can you make?

(b) Suppose you want exactly 8 toppings on your pizza. How many different pizzas can you make?

(c) What's going on here?

On the calculators, find $\binom{n}{r}$ by using the ${}_nC_r$ function. On the TI-84, type the first number, then find "nCr" under the MATH > PRB menu. On the TI-89, it is "nCr(n, r)" so you'll need to grab "nCr" under the MATH > PRB menu first. Or type it in. Teachers using both calculators have noted that "nCr" is often easier to find under CATALOG, since it is the first entry starting with the letter n.

(d) Explain why $\binom{n}{r} = \binom{n}{n-r}$.

16. Using the definition above, find $\binom{12}{0}$ and $\binom{12}{12}$. Explain the results using the concept of pizza toppings.

17. A 13-card hand is dealt from a standard 52-card deck. Find the probability that this hand contains exactly 3 aces and exactly 2 kings (which means exactly 8 of the rest. . .).

Tough Stuff

18. What's the *average* (mean) length of car used when you make all the trains of length 5? Is there a general rule at work here? Can you justify it?

19. In a coin-flipping game, you flip a fair coin (heads or tails) ten times. If you flip heads twice in a row at any point during the game, you lose. Find the probability that you win at this game.

20. A binary string of length 12 has twelve digits: all are ones or zeros. One example is

Jennifer asked this question during yesterday's session.

011010011100

How many 12-digit binary strings...
(a) ... do not start or end with a 1, *and*
(b) ... do not include any two consecutive ones?

4 *Training Day*

1. Suppose you have an unlimited supply of only rod lengths 3, 4, and 8. Find the total number of ways to make a train of length 14, using any method you like.

2. Suppose you have an unlimited supply of only rod lengths 4 and 7. Are there any train lengths you *can't* make? Which ones?

3. If you were allowed to write the letters of the word MINIMUM in any order, how many total different seven-letter "words" could you make?

One of these "words" is INMUMIM.

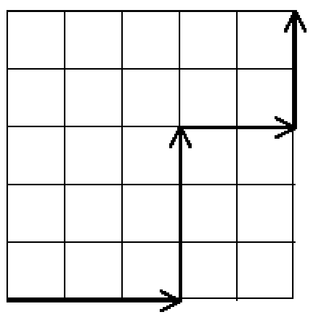
4. One way to make a train of length 14 is to use three whites (1-length rods), two reds (2-length rods), one green (3-length rod), and one purple (4-length rod). How many different-looking trains of length 14 could you make using these specific rods?

Neat and Useful Stuff

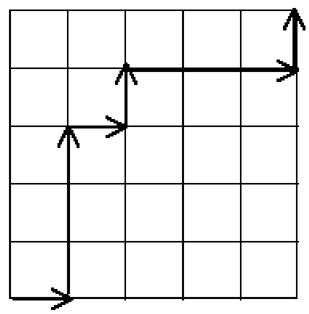
5. Using either factorial or "choose" notation, write an expression for the number of ways to reorder the letters of 70's singing sensation ABBA.

6. Repeat problem 5 for 80's singing sensation BANANARAMA.

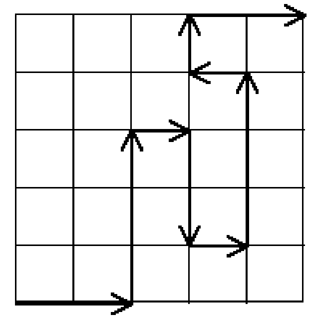
Ms. D'Amato likes to take a different route to work every day. She will quit her job the day she has to repeat her route. Her home and work are pictured in the grid of streets below.



A valid trip



Another valid trip



Not a valid trip

7. If Ms. D'Amato never backtracks (she only travels north or east), how many days will she work at this job?

8. How many more days can Ms. D'Amato work if she moves two blocks further away? Does it matter in which direction she moves?

9. Hey, what's the sum of the numbers in the fifth row of Pascal's Triangle? (Okay, so it's not a hard question.)

The fifth row of Pascal's Triangle starts 1, 5, . . .

10. Hey, what's the sum of the *squares* of the numbers in the fifth row of Pascal's Triangle? (See, a little harder.)

11. There are 12 students on a committee; 8 are juniors, and 4 are seniors. Determine the number of ways of forming the following subcommittees.

(a) A subcommittee of 5 juniors.
(b) A subcommittee of 3 seniors.
(c) A subcommittee of 6 students.
(d) A subcommittee of 3 juniors and 2 seniors.

12. A jar contains 3 white balls and 4 red balls. 2 balls are drawn. Find the probability that both balls are the same color. It may help you to...

(a) Find the number of ways to draw 2 balls from the jar.
(b) Find the number of ways to draw 2 white balls.
(c) Find the number of ways to draw 2 red balls.

13. Find P(same color) from problem 12 using a tree diagram. Which method do you prefer?

$P(A)$ means the probability of event A happening. Here, the "event" is drawing two balls of the same color.

14. (a) A committee of three is chosen from 3 Republicans and 4 Democrats. What is the probability that the committee consists of 2 Republicans and 1 Democrat?

(b) Verify your result in (a) using a tree diagram, where the committee members are selected one at a time.

Well... assume for the sake of this problem that the selection is random.

Tough Stuff

15. With 9 people, committees can be picked with as few as 0 people and as many as 9. Give a convincing argument (or, to use the vernacular, *prove*) that the total number of committees that can be formed with an odd number of members is the same as the number of committees with an even number of members.

16. With 8 people, committees can be picked with as few as 0 people and as many as 8. Give a convincing argument that the total number of committees that can be formed with an odd number of members is the same as the number of committees with an even number of members.

17. How many odd numbers are there in the 100th row of Pascal's Triangle?

5 *Trains of Thought*

The Leibniz Traingle... uh, Triangle... is given by

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & \frac{1}{2} & & \frac{1}{2} & & & \\
 & & \frac{1}{3} & & \frac{1}{6} & & \frac{1}{3} & & \\
 & \frac{1}{4} & & \frac{1}{12} & & \frac{1}{12} & & \frac{1}{4} & \\
\frac{1}{5} & & \frac{1}{20} & & \frac{1}{30} & & \frac{1}{20} & & \frac{1}{5}
\end{array}
$$

where each entry is the sum of the two numbers *below* it, and the initial and final entries in the nth row are $\frac{1}{n+1}$ (the single 1 at the top is referred to as the "zeroth" row, as it is in Pascal's Triangle).

1. Write down the next two rows of the Leibniz Triangle. Describe how you did it.

2. Consider the sequence of numbers along the second diagonal:

$$\frac{1}{2}, \frac{1}{6}, \frac{1}{12}, \frac{1}{20}, \frac{1}{30}, \frac{1}{42}, \ldots$$

(a) Derive a formula for this sequence.

(b) Can you *prove* why your formula in (a) has the form that it does? (Hint: Think about *how* you computed these numbers on the triangle.)

(c) Add the numbers in this sequence:

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \frac{1}{42} + \cdots$$

Does this series converge? If so, to which number and why?

3. Find some other patterns in the Liebniz Triangle. Don't spend more than 15 minutes on this problem, but spend at least 5.

Non-Useful Stuff

4. How many ways are there to use white and red Cuisenaire rods to build a cube, two units on a side? You are allowed to place red rods in any direction, including vertically.

This is a very nice problem, but it does not have any particular connection to later topics.

Neat and Useful Stuff

5. Without a calculator, expand $(h+t)^5$.

6. You flip a coin five times. How many ways are there to flip two heads and three tails?

7. How many trains of length 24 are made with exactly 5 rods of length 2, 3 rods of length 3, and 1 rod of length 5?

8. How many "words" can be made from ARTMABBOTT?

One such "word" is RATTATBOMB.

9. If you randomly rearrange the letters in ARTMABBOTT, find the probability that...
(a) ... the first letter is T.
(b) ... the first *two* letters are T.
(c) ... the first four letters are T.
(d) ... the letters spell out BRATATTOMB?

10. Find all train lengths you *can't* make if you only have rods of lengths 3 and 8.

11. Find all train lengths you *can't* make if you only have rods of lengths 3 and 9.

12. A bag contains 16 T-shirts of which 6 have red stripes. Two T-shirts are randomly drawn from the bag.
(a) Find the probability that neither T-shirt has red stripes.
(b) Verify your result using a tree diagram.
(c) Another bag contains 8 T-shirts of which 3 have red stripes. Is the probability of drawing two non-striped shirts the same for this bag? Why?

13. In a box of 12 batteries, it is known that 5 are dead. Four batteries are selected at random. Find the probability that…

(a) Exactly one dead battery is selected.

(b) All four of the selected batteries are dead.

(c) *At most two* of the selected batteries are dead.

14. A five-card hand is dealt from an ordinary deck of 52 cards. Find the probability of each hand. Express your answer in symbolic form.

(a) Four aces.

(b) Four of a kind. (Aces, or Kings, or twos, or… how many are there?)

(c) Exactly two hearts.

(d) At least three 10's.

(e) Three hearts and two spades.

(f) Three of one suit and two of another suit. (An example of this would be three hearts and two spades.)

15. There are two jars. The first jar contains 3 red, 2 blue, and 4 white marbles. The second jar contains 4 red, 3 blue, and 2 white marbles. A jar is chosen at random and 3 marbles are drawn. Find the probability that the three marbles are:

(a) all white

(b) all red

(c) all blue

(d) all the same color

Tough Stuff

16. In row 7 of Pascal's Triangle, the numbers 7, 21, and 35 appear consecutively. Interestingly, these three numbers are in arithmetic sequence. Does this ever happen again? If so, find the next three times it happens. If not, prove it can't happen again.

17. In poker, there is a structure of hand strength (what beats what). Here it is, from best to worst:

- Straight flush: five cards of consecutive rank and suit. For example, **7c 8c 9c Tc Jc** is a straight flush.
- Four of a kind: four cards of the same rank. For example, **7h 7c 7d 7s Ad** is a four of a kind hand.

One way to write suits is as a lowercase letter. In this problem, 7h means the seven of hearts, Tc means the ten of clubs, Qs means the queen of spades, and Ad means the ace of diamonds.

- Full house: three of one rank, and two of another. For example, **2c 2d 2h Jh Js** is a full house ("twos over jacks").
- Flush: five cards of same suit but not consecutive ranks. For example, **Tc Jc Qc Kc 5c** is a flush (all clubs) and very close to a straight flush.
- Straight: five cards of consecutive ranks but not the same suit. For example, **Ac 2d 3s 4c 5h** is a straight, ace through five.
- Three of a kind: Three cards of same rank, other cards of different ranks. For example, **Ks Kc Kd 6h 3s** is a three of a kind hand.
- Two pair: two pairs of different ranks, and one other card. For example, **6c 6s Td Ts Ks** is a two pair hand.
- One pair: two cards of same rank, other random cards. For example, **Ac Ah 3s 4d 5s** is a one pair hand.
- High card: None of the above. For example, **As Ks Qs Jc 9s** is a high card hand, since it has no pairs and is not a straight or flush.

A friend refers to a full house as an "Uncle Jesse" in honor of the ABC sitcom. Oy.

Aces can be high or low (so TJQKA is a straight) but no wraparound is allowed (so QKA23 is *not* a straight).

So, the question: if you're dealt five cards, how many ways are there to get each of these things? Is the hand strength correct, from hardest to easiest? For example, is it really harder to get three of a kind than to get two pair?

18. What, you want another problem? Okay. Prove that if p is a prime number, then $\binom{pa}{pb}$ has the same remainder as $\binom{a}{b}$ when you divide by p. For example, take $\binom{5}{2}$. Multiply the top and bottom by any prime (say, 7), and you'll get something with the same remainder when you divide by that prime. Using notation, you'd say $\binom{5}{2} = \binom{35}{14}$ modulo 7.