6 Trains, Planes, and... Binomials?

1. Expand (i.e., write without parentheses) each of the following.

(a) $(a+b)^2$
(b) $(a+b)^3$
(c) $(a+b)^4$
(d) $(a+b)^5$

2. What patterns do you observe in Problem 1? Use some triangle thing we passed out on Friday to expand $(a+b)^6$ without actually multiplying everything out.

3. Use a TI-89 to expand $(0.25r + 0.75w)^5$.

Check it out, it's a binomial...

4. You take an exam in Japanese with five multiple-choice questions. Each question has four possible answers, and one is right. The only problem is — you don't know any Japanese, so you're stuck making complete and utter random guesses.

(a) Find the probability of getting all five questions right.
(b) Find the probability of getting all five questions wrong.
(c) Find the probability of getting exactly two right.
(d) Is it more likely for you to get two questions right, or three questions right? Explain how you know.

5. On a ten-question true-or-false test, how many different *ways* are there to answer the test and get exactly seven questions right? Is there a notation for this?

6. Use some triangle thingy to find the number of different ways there are to answer a ten-question true-or-false test and get at least seven questions right.

7. What is the sum of the numbers in the 10th row of Pascal's Δ? How is this related to a ten-question true-or-false test?

Neat Stuff

8. Use the problems earlier in this set to expand $(M+N)^4$ where $M=2d$ and $N=7$. In other words, expand $(2d+7)^4$ without a calculator, but we ask it the weird way for a reason.

9. Does the line $4x+7y=22$ intersect any lattice points in Quadrant I of the coordinate plane?

Say again? *Lattice point*: a point with integer coordinates, like $(5, 11)$. *Quadrant I*: the zone where both x and y are positive.

10. For what positive integers N does the line $4x+7y=N$ *not* intersect any lattice points in Quadrant I? (For the purposes of this problem, a point on either axis *is* considered to be part of Quadrant I.)

We said in the title there'd be problems about planes...

11. You can calculate the difference of two cubes if you want to. Come on, it's fun:

$$\begin{aligned} 3^3-2^3 &= 27-8=19 \\ 4^3-3^3 &= 64-27=37 \\ 5^3-4^3 &= 125-64=61 \end{aligned}$$

Starting with $1^3-0^3=1$, find the sum of the first 100 differences of cubes. (The last one is 100^3-99^3.)

12. The function $f(x)=x^3$ is given below. For each output, find the *common difference* between consecutive inputs.

The notation for this is the Δ operator.

In: x	Out: $f(x)$	Δ
0	0	1
1	1	7
2	8	
3	27	
4	64	
5	125	91
6	216	
7	343	

13. Continue taking common differences for $f(x)=x^3$ until a constant value is found:

Input	Output	Δ	Δ^2	Δ^3
0	0	1	6	
1	1	7		
2	8			
3	27			
4	64			
5	125	91		
6	216			
7	343			

14. Holly sets you up with a whole lot of red rods (length 2).

(a) Find all the ways to make a 4-by-2 rectangle using red rods. A 4-by-2 rectangle has length 4 and width 2 only; no 2-by-4 rectangles qualify!

Rods may be arranged horizontally or vertically.

(b) Complete this table that gives the number of ways to use red rods to make n-by-2 rectangles for increasing values of n:

Like we said in the title...

Length: n	1	2	3	4	5	6	7	8
# Ways								

Tough Stuff

15. You've got an unlimited supply of green rods (length 3). Find the number of ways to make a 12-by-3 rectangle using only the green rods.Generalize to an unlimited supply of rods of length r, making an n-by-r rectangle using only rods of length r.

Rods may be arranged horizontally or vertically, again.

16. You're standing on the edge of a pool, facing away from it, and holding a bag with 4 white balls and 4 red balls. You pick a ball without replacement. If it's a white ball, take a step forward. If it's a red ball, take a step back (into the pool, sadly). If you survive, draw another ball and keep going until either

Time for Tough Stuff to get real tough. Good luck!

(a) ... you draw all the balls, or

(b) ... you're in the pool.

Find the number of different ways you could draw all the balls without entering the pool. Generalize to n balls of each color, or, if you prefer, colour.

7 *Changing Tracks...*

1. Spend 15 minutes revisiting the Simplex Lock Problem from Day 1. Use what you've learned to try and make some more progress toward a solution, or toward a different method if you've already found one.

2. Find the first five powers of 99, and explain what is happening using the Binomial Theorem or Pascal's Triangle... which are basically the same thing.

3. What's the sum of the numbers in the 8th row of Pascal's Triangle?

4. Each number in Pascal's Triangle is the sum of the two numbers above it. Use this to explain why the sum of the numbers in a row of Pascal's Triangle is a power of 2.

5. Suppose you want to make all the trains of length 3, but not all at the same time. You want to make them one at a time. How many of each car do you need? Well, here are the trains:

1–1–1,   1–2,   2–1,   3

You need three 1-cars, one 2-car (because any given train only uses one of them), and one 3-car.

How many cars (and which ones) do you need on your desk to make all the trains of length 4, doing it one train at a time? Now, suppose you want to make all the trains of length 5, one at a time. What do you need to *add* to the pile on your desk so you can do it? Then how many cars do you need to add to the pile in order to make all the trains of length 6? Generalize to length n: How many new cars do you need to add to a pile that lets you make all trains of length $n-1$ in order to get a pile that lets you make all trains of length n?

Neat Stuff

6. Use the Binomial Theorem to prove that

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Hint: Good choices of a and b in $(a+b)^n$ will get the job done.

This says that the sum of all the "choose" numbers $\binom{n}{k}$, as k goes from 0, 1, 2, ..., up to n, is 2^n.

7. Let $S = \{a_1, \ldots, a_n\}$ be an n-element set.
(a) How many 3-element subsets does S have? (Assume $n \geq 3$.)
(b) How many k-element subsets does S have? (Assume $n \geq k$.)

8. Let S be the set thing again from problem 7. How many *total* subsets are there? Here, k can be any number from 0 to n. See if you can do this problem two different ways.

As George Clooney once said, "You're either in or out."

9. When you expand $(3x^2 + \frac{5}{x})^6$, there is a constant term. What is it?

10. You roll a die five times.
(a) Use the Binomial Theorem (or expansion) to find the probability that you roll a six at least three times.
(b) What is the probability that you roll a six no more than twice?

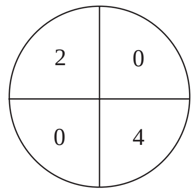
11. Here's a table for the function $f(x) = 2x^2 + 3x - 5$. Complete its common differences to Δ^3.

Input	Output	Δ	Δ^2	Δ^3
0	-5			
1	0			
2	9	13		
3	22			
4	39			
5	60			
6	85			
7	114			

12. Repeat problem 11 for each of these functions. Try to find some conjectures!

(a) $a(x) = 2x^2 - 10x + 8$	(d) $d(x) = 2x^3 + 5$
(b) $b(x) = 3x^2 + 12$	(e) $e(x) = 5x^3 - 12x^2 - x$
(c) $c(x) = -x^2 - 10x - 4$	(f) $g(x) = -7x + 12$

13. A number spinner is marked with four numbers like this:



All the regions are equally likely to be landed on. If you spin the spinner three times, what is the most likely *sum* of the four numbers? What sum is the next most likely?

14. Use a TI-89 to expand $(2 + x^2 + x^4)^3$. So what?

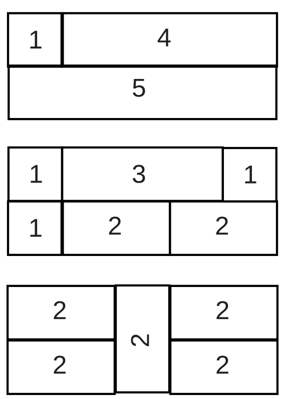
15. What is the most likely sum if you spin this spinner seven times?

Tough Stuff

16. All the numbers in the Liebniz Triangle were unit fractions. Isn't that weird? Prove it.

Prove the fact about the triangle, not that it is weird.

17. Suppose instead of $1 \times n$ rectangles, your trains were $2 \times n$ rectangles. How many $2 \times n$ rectangles are there? Here are just some 2×5 rectangles.



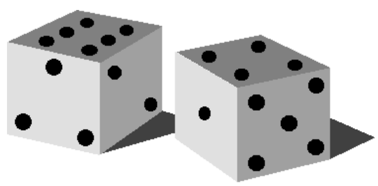
While you're at it, how many 3×5 rectangles are there?

This qualifies as "Useless Stuff".

8 *Count Those Exponents!*

In case you ever wondered when mathematics made a difference...

It's the **Difference Game**!



- Each player begins with 18 chips and a game board (each sold separately, void where prohibited, only for private club members).
- Players start by placing their chips in the numbered columns on their game boards. Boards are numbered from 0 to 5. The chips may be placed in any arrangement.
- Players take turns rolling the dice. The result of each roll is the difference between the number of dots on top of the two dice (the result of the roll shown above is 2).
- Each player who has a chip in the column corresponding to the result of the roll removes one chip from that column.
- The first player to remove all of the chips from his or her game board is the winner.

You could put your chips all in one column... or another column... or maybe that's not so smart.

1. Is there a winning strategy for the Difference Game?

2. Make a bar graph to record the results of twenty rolls of the dice.

3. Find the theoretical distribution of differences in this game. How could you use that to determine your starting strategy?

4. Use a TI-89 to expand:

$$(x+x^2+x^3+x^4+x^5+x^6)(x^{-1}+x^{-2}+x^{-3}+x^{-4}+x^{-5}+x^{-6})$$

Any thoughts?

5. (a) Find a polynomial that represents the distribution for a six-sided die.
(b) What do you get when you square that polynomial?

Look! There it is! Where? Over there!

Neat Stuff

A *set* is a grouping of numbers, like $\{1, 2, 3\}$. A *subset* is a grouping of numbers that may or may not contain any of the original set: $\{1, 3\}$ is a subset (and it's the same subset as $\{3, 1\}$). One subset contains all the elements, and one subset contains none of them: the notation, $\{\}$, is called the *empty set*.

6. (a) How many subsets of $\{1, 2, 3\}$ have exactly two elements?
(b) How many subsets of $\{1, 2, 3\}$ are there?
(c) How many subsets of $\{1, 2, 4, 8, 16\}$ have exactly three elements?

7. You can *partition* a set of numbers into non-empty subsets. For example, the set $\{1, 2, 3\}$ can be partitioned into two subsets: $\{1, 3\}$ and $\{2\}$ (which is the same as $\{2\}$ and $\{1, 3\}$). Or, it can be partitioned into two other subsets: $\{1, 2\}$ and $\{3\}$. It can even be partitioned into 1 or 3 subsets, though not in particularly exciting ways.
(a) How many total ways are there to partition $\{1, 2, 3\}$ into two subsets?
(b) How many total ways are there to partition $\{1, 2, 3, 4\}$ into two subsets?
(c) How many total ways are there to partition $\{1, 2, 3, 4, 5\}$ into two subsets?
(d) What's up with that?

What, you want this question in English? Okay, you've got four different-length Cuisinaire rods, and you want to break them up into groups so each group has at least one. How many ways are there to do this? Well, it depends on how many groups you're breaking it into...

8. Complete this table, with the number of elements as rows and the number of subsets as columns. Move, move, move!

	1	2	3	4	5
1		–	–	–	–
2			–	–	–
3	1	3	1	–	–
4		7			–
5					

9. What is the coefficient of x^{27} in the expansion of

$$\left(1 + x^5 + x^{10} + x^{15} + x^{20} + x^{25} + x^{30}\right)\left(1 + x^8 + x^{16} + x^{24} + x^{32}\right)$$

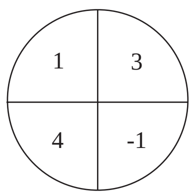
Feel free to use a TI-89, but think about how you might do without it.

10. Ben is run over by a train whose cars are either 5 meters or 8 meters long. What numbers are *not* possible as the total length of this train?

Assume all the stuff you need to. Yes, this is a vaguely repeated question!

11. Sometimes the numbers in the nth row of Pascal's Triangle (other than the 1s on the ends) are all divisible by n. When does this happen? Can you explain why?

12. Here's a spinner.



You get to spin the spinner once, then roll a die. Find the probability of having a total of 7 on the spinner and... uh... random number cube. Can you do this with polynomials?

13. The game of problem 7 changes. Before the game starts, you can select the number of times you're going to spin the wheel before spinning. You spin the spinner n times, then roll a... hexahedral event generator... and you're looking for a total of 7 from all the spins and the roll. Using the TI-89, find the value of n that makes it *most* likely for you to win the game.

Man, I hope this makes a lick of sense.

Tough Stuff

14. You have an unlimited supply of train lengths 1, 5, and 10. How many trains of length 50 can you make? Can you do this faster than the "recursive way"?

15. How many different combinations are there for a six-button Simplex lock?

9 *Die, Another Day*

1. In 1884, Park City's post office only sold 3- and 5-cent stamps, and you were only allowed to buy up to 6 of any one stamp. Use this table to figure out what denominations of postage could be made in *more than one way*:

Now, Park City sells bottles of air, priceless art with nearly-priceless price tags, and bears that can be used as a nativity scene. It's come a long way!

Rows: # of 3c stamps
Cols: # of 5c stamps

	0	1	2	3	4	5	6
0					12		
1							33
2				19			
3							
4		13					
5							
6	18					43	

2. Okay, so we made the last one up. Anyway, multiply this out on the TI-89 and comment:

Mmm, mechanical...

$$\left(1+x^5+x^{10}+x^{15}+x^{20}+x^{25}+x^{30}\right)\left(1+x^3+x^6+x^9+x^{12}+x^{15}+x^{18}\right)$$

3. What is the sum of the coefficients of the big honking thing you multiplied to get in problem 2?

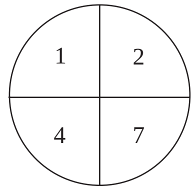
4. Using polynomials and the TI-89, find the probability that when you roll four dice, you get a sum that is less than 10.

5. Using polynomials and the TI-89, find the probability that when you roll four dice, you get a sum that is more than 18.

6. Using polynomials and the TI-89, draw a histogram for the number of ways to get any sum (from 4 to 24) with four dice rolls. Any thoughts on the shape?

A *histogram* is like a bar graph, but there aren't gaps between bars. Yesterday, Ben drew histograms, not bar graphs.

7. Here's a spinner.



It may help to think of the spinner as a 4-sided die with specific numbers on each side, instead of 1 through 6.

(a) Write a polynomial that can represent one spin.
(b) Find the most likely sum for three spins.
(c) Draw a histogram for the number of ways to get any sum with four spins. Any thoughts on the shape?

Neat Stuff

A *set* is a grouping of anything, like {Peg,Marta,Tony}. A *subset* is a grouping of stuff that may or may not contain any of the original set: {Peg,Tony} is a subset (and it's the same subset as {Tony,Peg}). One subset contains all the elements, and one subset contains none of them: the *empty set*, notated by {}.

8. (a) How many subsets of {Peg,Marta,Tony} have exactly two elements?
(b) How many subsets of {Cheryl,Judy,Remy} are there?
(c) How many subsets of {Jason,Lars,Gerry,Ryota,inanimate carbon rod} have exactly three elements?

9. You can *partition* a set into non-empty subsets. For example, the set {Marta,Tony,Peg} can be partitioned into two subsets: {Marta,Peg} and {Tony} (which is the same as {Tony} and {Peg, Marta}). Or, it can be partitioned into two other subsets in two other ways. It can even be partitioned into 1 or 3 subsets, though not in particularly exciting ways.
(a) How many total ways are there to partition a set of four people into exactly two non-empty subsets?
(b) How many total ways are there to partition a set of *five* people into exactly two non-empty subsets?
(c) Six?
(d) What's the deal? Why is this happening?

What, you want this question in English? The whole problem barely has any numbers in it! Okay, okay, try this: you've got four different-length rods, and you want to break them up into groups with no "empty" groups. How many ways are there to do this? Well, it depends on how many groups you're breaking it into...

10. A box contains balls marked $1, 2, 3, 4, \ldots, n$. Two balls are chosen. Find the probability that the numbers on the balls are consecutive integers.

Hint for problem 10**:** Try it with specific values of n first. Don't worry, you don't need polynomials for this, and this is "just for fun" (no later connections).

11. Complete this table, with the number of elements (people, numbers, whatever) as rows and the number of non-empty subsets as columns.

	1	2	3	4	5
1		–	–	–	–
2			–	–	–
3	1	3	1	–	–
4		7			–
5					

12. Continue the table of problem 11 until you find a recursive rule you could use to continue the table even further.

13. A local pizza restaurant whose building is shaped like a hut offers 18 toppings on their pizza. When you select a pizza, you can choose anywhere from zero to three toppings, *or* you may choose one of five specialty pizzas whose toppings are different and preset. How many different kinds of pizza can be ordered at this hut of pizza?

Just one pizza. Don't fret over large / small or the fact that the Cheese Lovers' Plus specialty pizzas allows for two topping suggestions. If you want, do that as a bonus, but... oh just do the problem already.

14. The numbers $\binom{n}{2}$ (the triangular numbers) have a polynomial rule. Find this rule and sketch a graph of this function for all real numbers (even though $\binom{n}{2}$ only makes sense for integer $n \geq 2$).

15. Repeat problem 14 for $\binom{n}{1}$, $\binom{n}{3}$, and $\binom{n}{4}$. Is something happening in general?

Tough Stuff

16. Kellie and Jessica are ordering pizza at Pizza Hut. Jessica suggests the "4 For All" pizza, which is actually four little pizzas with the same choices of toppings as in problem 13. An uncreative way is to select all four pizzas to be the same, but they could all be different, or... well there are some options there. Jessica shouts that the "4 For All" gives you more than six million topping options, and seems overly excited about the whole thing.
Is she right? Are there more than six million options here? How many options *are* there?

17. Could there be a connection between the table of problem 11 and the solution to the Simplex Lock Problem? Surely there couldn't possibly...

Tired of the Simplex Lock Problem yet? Come on, you know you are!

10 Could You Expand On That?

Psst: this set might be long! Some would say it's been expanded. Do what you can! *Don't worry about doing all the problems*!

Here's a note that, in hindsight, belonged on Day 1:

- *Whatever you do, do well.* Flying through the problem set helps no one, especially yourself—you're going to miss the big ideas that others are grabbing onto! There is more to be found in these problems than their answers.

1. Use the following table, and *not* a TI-89, to find

$(1+x^3+x^6+x^9+x^{12}+x^{15}+x^{18})(1+x^5+x^{10}+x^{15}+x^{20}+x^{25}+x^{30})$

The table has deliberately been left empty, since the authors do not trust themselves to fill in the table properly.

	1	x^5	x^{10}	x^{15}	x^{20}	x^{25}	x^{30}
1							
x^3							
x^6							
x^9							
x^{12}							
x^{15}							
x^{18}							

Any thoughts? (Other than, "Can I move on now?")

2. Is it possible to make 50¢ using only 8¢ and 11¢ stamps?

3. Say you were going to expand this:

"You were going to expand this."

$$(1 + x^8 + x^{16} + x^{24} + \cdots)(1 + x^{11} + x^{22} + x^{33} + \cdots)$$

Without expanding, figure out what the coefficient of x^{50} should be. What is the meaning of all this?

4. (a) Expand $(r + w + b)^5$. Sure, use that TI-89 this time.

(b) Suppose there are five jars, each containing a red marble, a white marble, and a blue marble. You take one marble from each jar. How many ways are there to pick up two reds, two whites, and one blue?

5. Find a polynomial that can be used to represent the value of a playing card from a deck of cards. Aces are worth one point, and the face cards (jack, queen, king) are each worth ten.

6. Suppose you have 7 people and you want to form a 3-member committee.

(a) How many such committees are there?

(b) Pick one of the 7 people, say John. Of all the possible committees, how many of them contain John?

Hint: Since John must take up a spot, there are 2 spots open. How many people can fill those 2 remaining spots?

(c) Of all the possible 3-member committees with 7 people, how many of them do *not* contain John?

(d) Explain why

$$\binom{7}{3} = \binom{6}{2} + \binom{6}{3}$$

(e) Give a "committee" proof of this identity that can be visualized in Pascal's Triangle (where $0 < k < n$):

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Describe how this is visualized in Pascal's Triangle.

Neat Stuff

7. Suppose we have the set

$$\{\text{Alicia}, \text{Bill}, \text{Claudia}\}$$

(a) List all the possible ways to partition this set into exactly two non-empty subsets.

"Partition" just means "break up".

(b) List all the possible ways to partition this set into exactly one non-empty subset. Yeah, this one's quick.

(c) Using your results from (a) and (b), derive all possible ways to partition the set

$$\{\text{Alicia}, \text{Bill}, \text{Claudia}, \text{Donna}\}$$

into exactly two non-empty subsets.

8. Ponder again the set

$$\{\text{Alicia}, \text{Bill}, \text{Claudia}\}$$

(a) List all the possible ways to partition this set into exactly three non-empty subsets.
(b) List all the possible ways to partition this set into exactly two non-empty subsets. Yes, again.
(c) Using your results from (a) and (b), derive all possible ways to partition the set

$$\{\text{Alicia}, \text{Bill}, \text{Claudia}, \text{Donna}\}$$

into exactly three non-empty subsets.

9. Let $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ denote the number of ways to partition a set of m people into k non-empty subsets. Hey, it's just notation.
(a) Using Problem 6, explain why

$$\left\{ \begin{matrix} 4 \\ 2 \end{matrix} \right\} = 2\left\{ \begin{matrix} 3 \\ 2 \end{matrix} \right\} + \left\{ \begin{matrix} 3 \\ 1 \end{matrix} \right\}$$

(b) Using Problem 7, find the relationship between $\left\{ \begin{matrix} 4 \\ 3 \end{matrix} \right\}$, $\left\{ \begin{matrix} 3 \\ 3 \end{matrix} \right\}$, and $\left\{ \begin{matrix} 3 \\ 2 \end{matrix} \right\}$.
(c) Find a recursive rule for $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$.

10. Hey, this table has been on the set for a couple of days. Complete the table, with the number of elements (people, numbers, whatever) as rows and the number of non-empty subsets as columns.

You know what *that* means...

	1	2	3	4	5	6
1		-	-	-	-	-
2			-	-	-	-
3	1	3	1	-	-	-
4		7			-	-
5						-
6						

11. On a 6-button Simplex Lock, find the number of combinations that use all six buttons and contain exactly 2 pushes. Is there any connection with the table above? Heck no! Okay, maybe there is. Find it.

In a *function*, all the elements from the *domain* (read: "inputs") map to some element of the *range* (read: "possible outputs"). Every input must lead to *some* output, but that doesn't mean that every output has to have inputs feeding it.

12. Here are some more lovely sets:

$$S = \{\text{Nick,Ellie,Megan,Lynda,Chris}\} \quad \text{and} \quad T = \{\text{first,second}\}$$

(a) How many functions are there with domain S and range T? Think about choice!
(b) How many functions are there with domain T and range S? The choices are different now!

13. A function from S to T is called *one-to-one* if no two elements in S get mapped to the same element in T.
(a) In Problem 11, how many functions from S to T are one-to-one?
(b) In Problem 11, how many functions from T to S are one-to-one?

If there are any, try making a list. If there aren't any, why not?

14. A function from S to T is called *onto* if every element in T gets hit by some element of S.
(a) In Problem 11, how many functions from S to T are onto?
(b) In Problem 11, how many functions from T to S are onto?

15. Complete this difference table for $y = x^5$:

Input	Output	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
0	0	1					
1	1	31	180				
2	32	211					
3	243						
4	1024						
5	3125						
6							
7							

16. In the table of problem 15, find the sum of the numbers *across* the row for input 0: that is, $0 + 1 + 30 + \ldots$. Hmmmm?

Tough Stuff

17. In the expansion of $(x + x^2 + x^3 + x^4 + x^5 + x^6)^4$, the coefficients look like

$$1, 4, 10, 20, 35, 56, 80, 104, 125, \ldots$$

i.e., the expansion looks like

$$x^{24} + 4x^{23} + 10x^{22} + 20x^{21} + \cdots$$

Can you find a similar sequence of numbers in the Pascal's Triangle? Where do the two sequences fail to match? What's going on here?!

18. Give a "committee" proof of the identity $(2 \leq k \leq n-2)$:

$$\binom{n}{k} = \binom{n-2}{k-2} + 2\binom{n-2}{k-1} + \binom{n-2}{k}$$

19. Suppose you have a 25 element set S and a 360 element set T.

(a) How many functions from S to T are there?

(b) How many functions from S to T are one-to-one?

(c) If you pick a function from S to T at random, what's the probability that it is *not* one-to-one?

20. Ten players are each dealt two cards from a fair deck of 52 playing cards. Find the probability that any one player is dealt a pair of aces, *and* a second player is dealt a pair of kings. It may be easier to find the probability that these events do *not* happen, and feel free to use decimals instead of exact fractions here.