# 11 *th Hour?*

**1**. Suppose there are 5 people.

(a) How many 2-member committees can you form?

(b) Find the number of $k$-member committees you can form for $k = 0, 1, 2, 3, 4,$ and 5.

(c) Using (b), find the *total* number of committees you can form using any of these five people.

Well, really there are more than 5 people, but there are 5 people to pick from in this problem.

**2**. Let the five people in Problem 1 be:

$$\{\text{Alicia}, \text{Bill}, \text{Claudia}, \text{Donna}, \text{Eugene}\}$$

Think about how you'd form a committee by including or excluding each person. For Bill (who is sitting on Capitol Hill, presumably), you'd decide to make Bill part of your law-making committee, or choose to veto him from joining.

(a) Using the line of reasoning above, explain why there are $2^5$ possible committees you can form.

(b) Use the information in these two problems to explain why

$$\binom{5}{0} + \binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5} = 2^5$$

(Don't just say that they're both equal to 32.)

(c) Use a "committee" proof to explain why the sum of the numbers in the $n$th row of Pascal's Triangle is $2^n$.

**3**. 14 people are eligible to join the Teen Girl Squad, and the head of the squad is selecting who can join.

(a) Explain why there are just as many 10-girl squad choices as 4-girl squad choices.

(b) Create a *mapping* that shows that each 10-girl squad corresponds in a one-to-one way with each 4-girl squad.

**4**. Suppose you have 10 people and you want to form a 4-player golf team.

(a) How many such teams are there? Order doesn't matter.

(b) Pick two of the 10 people, say John and Jen. Of all the possible 4-member teams, how many of them contain *both* John and Jen?

(c) Of all the possible 4-member teams, how many of them contain *neither* John nor Jen?

(d) How many of them contain John, but not Jen? Jen but not John?

(e) Explain why

$$\binom{10}{4} = \binom{8}{2} + 2\binom{8}{3} + \binom{8}{4}$$

(f) Describe what this statement says about Pascal's Triangle and about committees:

$$\binom{n}{k} = \binom{n-2}{k-2} + 2\binom{n-2}{k-1} + \binom{n-2}{k}$$

**5**. Suppose you have a group of 12 people and you want to form a committee of 7 people. Of the 7 people chosen, 4 will also be selected as members of a subcommitteee. Find two different ways to count the number of possibilites here.

What if the subcommittee was selected *before* the rest of the committee? What if the committee was selected first?

**6**. Hey, is this true? Can you explain it?

$$\binom{15}{8}\binom{8}{3} = \binom{15}{3}\binom{12}{5}$$

**7**. It's the *Birthday Problem*! What's the probability that at least two people in a room of 20 have the same birthday? What if there are 25 people in the room? How many people have to be in the room before the probability of two same-day birthdays is more than 0.5?

## Neat Stuff

**8**. Bob gives you two big wheels to spin. The first wheel has four numbers on it:

$$A = 1, 7, 9, 15$$



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The second wheel has five numbers on it:

$$B = 2, 8, 14, 26, 35$$

Bob tells you to spin each wheel once, and you'll win \$1,000 for each point you score above average. But first, you'll have to answer some questions:

(a) What *is* the average you'll score if you spin each wheel once?

(b) What's the probability you will win at least \$10,000 by playing this game?

(c) What's the average you'll score on the first wheel? the second wheel? Hm.

**9**. When finding a line of best fit, the **mean squared error** is calculated to judge how good the fit is:

- Find out how far away the data is from what it should be.
- Square that value.
- Add up all the squares, then divide by how many data values there are.

This "least squares method" is generally accepted as the way to test best fit lines. So what? Well, you can do the same thing with a set of numbers, by taking their differences from the mean. This is called the *mean squared deviation*, more commonly known as the *variance*.

(a) Hey, find the variance for set A from problem 8.

(b) Find the variance for set B from problem 8.

(c) Find the variance for tossing one die. (Okay, it won't be an integer this time.)

(d) Find the variance for the distribution of tossing *two* dice. Hm?

**10**. Here are some sets of stuff found at Cows.

$$F = \{\text{chocolate, vanilla, smoores, wowie cowie, cherry springer}\}$$

$$C = \{\text{sugar, waffle, cup}\}$$

(a) How many functions are there with domain $F$ and range $C$?

(b) How many functions are there with domain $C$ and range $F$?

**11**. (a) In Problem 10, how many functions from $F$ to $C$ are one-to-one? (i.e., no two elements in $F$ get mapped to the same element in $C$.)

(b) In Problem 10, how many functions from $C$ to $F$ are one-to-one?

**12**. (a) In Problem 10, how many functions from $F$ to $C$ are onto? (i.e., every element in $C$ gets hit by some element of $F$.)
(b) In Problem 10, how many functions from $C$ to $F$ are onto?

**13**. On a 5-button Simplex Lock, find the number of combinations that use all five buttons and contain exactly 3 pushes. What's going on here?

**14**. Suppose you have a 25 element set $S$ and a 360 element set $T$.
(a) How many functions from $S$ to $T$ are there?
(b) How many functions from $S$ to $T$ are one-to-one?
(c) If you pick a function from $S$ to $T$ at random, what's the probability that it is *not* one-to-one?
(d) Why would we ask this question?

### Tough Stuff

**15**. Use trains of length 1 and 2 to prove this fact about Fibonacci numbers:

$$F_{m+n} = F_n F_m + F_{n-1} F_{m-1}$$

$F_n$ means the nth Fibonacci number. The Fibonacci numbers go $1, 2, 3, 5, 8, 13, 21, 34 \ldots$. So, for example, $34 = 8 \cdot 3 + 5 \cdot 2$.

**16**. In general, prove that

$$\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$$

**17**. Give at least two proofs of the identity

$$r\binom{n}{r} = n\binom{n-1}{r-1}$$

**18**. Give at least two proofs of the identity

$$(r+1)\binom{n}{r+1} = (n-r)\binom{n}{r}$$


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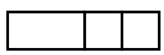
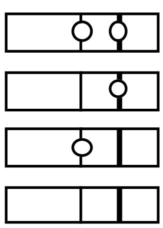
# 12 *th Night*

Trains!!! What, this again? Ben works as an arc welder on trains now, in an attempt to make train problems more difficult. You have already found the 8 ways to make trains of length 4. Now, imagine that the cars can be welded together or not. So now,



could be any of these four (the circle represents that the cars are welded together):

You can only weld where there is a break already.



**1**. List or build all the ways to arrange a train of length 1,2 and 3 using welding.

**2**. Fill in the following tables:

Oh good, no tables. I guess that's the end of *this* problem.

(a)

| Number of ways to make a train of length 1 using welding | |
|---|---|
| | 0 welds |
| 1car | |

D'oh.

(b)

| Number of ways to make a train of length 2 using welding | | |
|---|---|---|
| | 0 welds | 1 weld |
| 1car | | |
| 2 cars | | |

(c)

| Number of ways to make a train of length 3 using welding | | | |
|---|---|---|---|
| | 0 welds | 1 weld | 2 welds |
| 1car | | | |
| 2 cars | | | |
| 3 cars | | | |

**3**. Make a table as in problem 2 above for trains of length 4.

**4**. What might the first column and bottom row of the table for a train of length 5 with welding look like? What about the other entries?

**5**. Suppose you have a train of length 13 and you want to select a train that has 7 "gaps". Of the 7 gaps, 4 will also be selected to be welded. Find two different ways to count the number of possibilites here. Psst: see yesterday.

What if the welds were selected *before* the rest of the gaps? What if the gaps were selected first?

**6**. With a TI-89, expand $(x + y + z)^3$. Any relationship to the arc welding tables you made?

**7**. Find the sum of all the numbers in the arc welding table for trains of length 1, length 2, length 3, length 4. What's up with that?

**8**. Explain why the sum of the coefficients of $(x + y + z)^9$ is ______ without expanding and counting. Oh, dear, we seem to have left the number blank.

## Neat Stuff

**9**. Here are some sets that we've seen before:



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$$S = \{\text{Nick, Ellie, Megan, Lynda, Chris}\}$$

$$T = \{\text{first, second}\}$$

(a) As a function, "Nick" maps either to "first" or "second". How many choices is this?
(b) How many choices are there for Ellie's output?
(c) How many functions are there from $S$ to $T$? A *function* has exactly one output for every input.
(d) A function from $S$ to $T$ is *onto* if every element in $T$ gets hit by some element of $S$. One way to do this is to have the element "first" be hit by 2 elements of $S$ and "second" be hit by 3 elements of $S$. How many functions from $S$ to $T$ have "first" as output twice and "second" as output three times?
(e) Find the total number of onto functions from $S$ to $T$.

We can have "first" be hit by 4 elements of $S$ and "second" by 1 element of $S$, and so on...

**10**. On a 5-button Simplex Lock, find the number of combinations that use all five buttons and exactly 2 pushes. Hm?

**11**. Yesterday, Bob showed you the two wheels

$$A = 1, 7, 9, 15$$

$$B = 2, 8, 14, 26, 35$$

(a) Do problem 8 from yesterday if you haven't gotten the chance yet.
(b) Do problem 9 from yesterday if you haven't gotten the chance yet.
(c) The *mean squared error*, or *variance*, of a data set is based on the squares of data. If the numbers in $B$ were "inches", then the variance is based on square inches! So, the square root of the variance would be in "inches" again (a good thing). This measure, the square root of variance, is called the *standard deviation*. Find the standard deviations for sets $A$ and $B$.

**12**. Find the variance and standard deviation for the set of the 20 possible *sums* you get from spinning the two wheels $A$ and $B$. Hm?

**13**. Find the variance and standard deviation for the distribution when rolling...
(a) ... one die.

(b) ... two dice.
(c) ... three dice.
(d) ... four dice.
(e) ... twenty-five dice.

Jeez, 25 dice? Why pick such an odd number?

**14**. Find the variance and standard deviation for flipping a coin (0 = tails, 1 = heads).
(a) Why does the term "standard deviation" make sense here (once you know what its value is)?
(b) Use what you've learned and/or guessed to find the standard deviation for flipping two coins, four coins, 100 coins.

## Tough Stuff

**15**. A "Bernoulli trial" is a one-time event that has probability $p$ and alternate probability $q = 1 - p$. The formula for the standard deviation of $n$ Bernoulli trials is

$$\sigma = \sqrt{npq}$$

Explain why this formula is valid.

You might have more fun using variance and not standard deviation here.

**16**. (a) Explain why

You're choosing 3 people out of 8. Split the 8-person set into two equal pieces.

$$\binom{8}{3} = \binom{4}{0}\binom{4}{3} + \binom{4}{1}\binom{4}{2} + \binom{4}{2}\binom{4}{1} + \binom{4}{3}\binom{4}{0}$$

(Don't just say that they're both equal to 56.)
(b) Give a "committee" proof of this identity:

$$\binom{2n}{r} = \sum_{k=0}^{r} \binom{n}{k}\binom{n}{r-k}$$

**17**. For this problem, you'll need a copy of the Pascal's Triangle with at least
15 rows.
(a) Square each entry in the 4th row of the triangle and add them up. Can you
find this sum anywhere in the triangle?
(b) Repeat (a) using rows 5, 6, and 7 of the triangle. Any conjectures?

# 13 *th Floor*

The 13th floor may be numbered 12, 13, or 14, depending on where you're traveling...

**1**. Using a TI-89, find the number of ways to make a train of length 7 with 5 cars and 2 welds. Remember, a train of length 7 will have 6 "gaps".

**2**. What's the sum of the coefficients of $(x+y+z)^6$, and what's it got to do with trains and welds and all that good stuff?

**3**. Wink shows you two coins with numbers on them.

$$A = 1, 7 \text{ and } B = 5, 13$$

(a) Find the variance and standard deviation for flipping coin $A$.
(b) Find the variance and standard deviation for flipping $B$.
(c) Find the variance and standard deviation for the set of the possible *sums* you get from flipping each of $A$ and $B$. Hm?
(d) What if you flipped *two* of each coin then added the sum of all four coins? Would the standard deviation double?

**4**. Find the variance and standard deviation for the distribution when rolling...
(a) ... one die.
(b) ... two dice.
(c) ... three dice.
(d) ... four dice.

(e) ... forty-nine dice.

How would the variance relate to the original variance?... the standard deviation?

**5**. Find the variance and standard deviation for flipping a coin (0 = tails, 1 = heads).

(a) Why does the term "standard deviation" make sense here (once you know what its value is)?

(b) Use what you've learned and/or guessed to find the standard deviation for flipping two coins, four coins, 100 coins.

**6**. Here are some not-so-subtly named sets:

$$B = \{\text{button1, button2, button3, button4, button5}\}$$

$$P = \{\text{push1, push2, push3}\}$$

(a) How many functions are there from $B$ to $P$?

Each of the five inputs has three possible outputs. 3 choices, then 3 choices, then...

(b) A function from $B$ to $P$ is *onto* if every element in $P$ gets hit by some element of $B$. One way to do this is to have the element "push1" be hit by 2 elements of $B$, "push2" be hit by 1 element of $B$, and "push3" by hit by 2 elements of $B$. How many such onto functions are there (with exactly these choices of outputs)?

(c) Find the total number of onto functions from $B$ to $P$. Remember, there must be at least one match for every output. How can 3 pushes be related to trains? No, there's no welding!

**7**. On a 5-button Simplex Lock, find the number of combinations that use all five buttons and exactly 3 pushes. Okay, maybe the names $B$ and $P$ and their contents from problem 6 gives this one away!

**8**. Find the largest Simplex Lock you've worked with. Find (from your notes or a tablemate) the number of combinations on that-size lock, and the number of combinations that use all the buttons. Hm?

## Tedious but Vaguely Interesting Stuff

**9**. Complete the difference table for $y = x^n$ for $n = 2, 3, 4, 5$. (See Day 10, Problem 15 for the set-up). For each table, find the sum of the numbers *across* the row for input 0. Any thoughts?



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### Neat Stuff

**10**. Suppose we have 4 people in a class studying Set:

$$\{\text{Alicia, Bill, Claudia, Donna}\}$$

(a) List all the possible ways to break these guys up into three groups so the groups are noticably different: so $A, B, C, D$ is the same as $C, D, A, B$.
(b) List all the ways you could break these guys up into two groups.
(c) Marzipan walks in the room and these four are working in three groups. If a teacher wants there to be three groups for the five people, how many choices do they have about where to put poor late Marzipan?
(d) Eugene walks in the room and these four are working in two groups. If a teacher wants there to be three groups for the five people, how many choices do they have about where to put poor late Marzipan?

**11**. You may remember that we used $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ to mean the number of ways to split up $n$ people into $k$ non-empty subsets. Or, you probably don't. Well, there it is again.
(a) Using problem 10, explain why

$$\left\{ \begin{matrix} 5 \\ 3 \end{matrix} \right\} = 3 \left\{ \begin{matrix} 4 \\ 3 \end{matrix} \right\} + \left\{ \begin{matrix} 4 \\ 2 \end{matrix} \right\}$$

(b) Use a "grouping" argument to explain why this is true no matter how large the groups are:

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = k \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\} + \left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\}$$

Think of Marzipan walking into a class of 100 people divided into 8 groups. What options does the teacher have for poor late Marzipan?

**12**. If you haven't before, complete the following table. The number of elements (people, numbers, whatever) are the rows and the number of non-empty subsets are the columns.

| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | | - | - | - | - |
| 2 | | | - | - | - |
| 3 | 1 | 3 | 1 | - | - |
| 4 | | 7 | | | - |
| 5 | | | | | |

Psst! Problem 11 tells you what you need to know to keep on building the table. Use it!

**13**. Nick, Ellie, Megan, and Lynda are flying home from Park City. Each person can fly either first-class or coach. Thus, one possible seating arrangement is

| first-class | coach |
|---|---|
| Nick, Ellie | Megan, Lynda |

List all possible seating arrangements, provided that at least one person must fly first-class and at least one person must fly coach.

**14**. Problem 13 is like problem 10 (the groupings) {Nick, Ellie, Megan, Lynda} can be order-partitioned into two subsets: {Nick, Ellie} and {Megan, Lynda} (which is *different* from {Megan, Lynda} and {Nick, Ellie}).

(a) How many total ways are there to order-partition a set of three people into exactly two non-empty subsets?

(b) How many total ways are there to order-partition a set of four people into exactly two non-empty subsets?

(c) How many total ways are there to order-partition a set of five people into exactly two non-empty subsets?

**15**. Complete this table, with the number of elements (people, numbers, whatever) as rows and the number of non-empty subsets as columns.

| | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 1 | - | - | - | - | - |
| 1 | - | 1 | - | - | - | - |
| 2 | - | | | - | - | - |
| 3 | - | | 6 | | - | - |
| 4 | - | | | | | - |
| 5 | - | | 30 | | | |

**16**. Find the sum of the fifth row of the table above. Any thoughts?

**17**. Continue the table from Problem 15 until you find a recursive rule you could use to continue the table even further.

**18**. How are the tables in Problem 12 and Problem 15 related?

## Tough Stuff

**19**. Suppose you have 7 apples and 5 bananas, and you wish to pick 4 pieces of fruit.



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(a) In how many ways can you take no apple and four bananas?
(b) In how many ways can you take one apple and three bananas?
(c) In how many ways can you take two apples and two bananas?
(d) In how many ways can you take three apples and one banana?
(e) In how many ways can you take four apples and no banana?

**20**. Prove Vandermonde's Identity: If $m$, $n$, and $r$ are non-negative integers,

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}$$

**21**. Glenn tells you about a "verrry simmmple" concept. He says you can pick any three consecutive integers and their product is a multiple of 6. Pick any four consecutive integers and their product is a multiple of 24. Prove it!

**22**. The expansion of $(a+b)^n$ leads to Pascal's Triangle. So... in the expansion of $(x+y+z)^n$ (for different $n$), what will the largest coefficient be? When will it be unique? When it isn't unique, how many repeats are there?

# 14 th Hole

**1**. The newly created American Dodgeball League (ADL) has five teams which must be placed into three divisions. The teams are: Giants, Eagles, Rams, Panthers, and Jets. The divisions are: East, Central, and West. For example, the teams can be divided as follows.

| East | Central | West |
|---|---|---|
| Giants, Panthers | Eagles | Rams, Jets |

Each division must contain at least one team. In how many different ways can the teams be divided?

**2**. A ridiculously small class has 5 people:

{Alicia, Bill, Claudia, Donna, Ziggy}

Their teacher breaks them up into the following three groups:

- Alicia, Ziggy
- Bill
- Claudia, Donna

Each group must give a presentation on one of the following topics: Algebra, Geometry, and Combinatorics. And no two groups can pick the same topic. In how many different ways can the three topics be assigned to the three groups?

**3**. Remember that $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ denotes the number of ways to partition a set of $n$ people into $k$ non-empty subsets.

(a) Find (or look up from your previous work) the value of $\left\{ \begin{matrix} 5 \\ 3 \end{matrix} \right\}$.

(b) Now, let $\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle$ denote the number of ways to *order-partition* a set of $n$ people into $k$ non-empty subsets. (e.g., we count {Nick, Ellie} and {Megan, Lynda} to


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be *different* from {Megan, Lynda} and {Nick, Ellie}.) Find the value of $\left\langle {5 \atop 3} \right\rangle$.

(c) How are $\left\{ {5 \atop 3} \right\}$ and $\left\langle {5 \atop 3} \right\rangle$ related?

(d) How are $\left\{ {n \atop k} \right\}$ and $\left\langle {n \atop k} \right\rangle$ related?

Who is a bowl? Who is a cone? What??

**4**. Yesterday's set included this recursive rule for $\left\{ {n \atop k} \right\}$:

$$\{n, k\} = k\{n-1, k\} + \{n-1, k-1\}$$

And that rule led to this table (the number of elements are the rows and the number of non-empty subsets are the columns):

| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 1 | - | - | - | - |
| 2 | 1 | 1 | - | - | - |
| 3 | 1 | 3 | 1 | - | - |
| 4 | 1 | 7 | 6 | 1 | - |
| 5 | 1 | 15 | 25 | 10 | 1 |

Complete the following table for $\left\langle {n \atop k} \right\rangle$:

| | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 1 | - | - | - | - | - |
| 1 | - | 1 | - | - | - | - |
| 2 | - | | | - | - | - |
| 3 | - | | 6 | | - | - |
| 4 | - | | | | - | |
| 5 | - | | 30 | | | |

**5**. Make a difference table for $y = x^4$ (to fourth differences) for $x = 0$ through 6. When you're done, what numbers appear across the top row?

## Neat Stuff

**6**. Consider the set

These three are getting an awful lot of attention...

$$\{\text{Alicia, Bill, Claudia}\}$$

(a) List all the possible ways to *order-partition* these guys into exactly two non-empty subsets.

(b) List all the possible ways to order-partition these guys into exactly one non-empty subset. Yeah, there aren't too many, are there?

(c) Now Gaylord shows up and wants to join in the fun. List all the possible ways to order-partition

$$\{\text{Alicia, Bill, Claudia, Gaylord}\}$$

into exactly two non-empty subsets. You might... just might... want to use the lists you made in the other problems. If you don't do it that way, stop and look back when you're done with the list.

**7**. Again! These same people!

$$\{\text{Alicia, Bill, Claudia}\}$$

(a) List all the possible ways to *order-partition* this set into exactly three non-empty subsets.

(b) List all the possible ways to order-partition this set into exactly two non-empty subsets. We love asking the same question over and over.

(c) Using your results from (a) and (b), derive all possible ways to order-partition the set

$$\{\text{Alicia, Bill, Claudia, Phineas}\}$$

into exactly three non-empty subsets.

**8**. (a) Using Problem 6, explain why

$$\left\langle \begin{matrix} 4 \\ 2 \end{matrix} \right\rangle = 2\left\langle \begin{matrix} 3 \\ 2 \end{matrix} \right\rangle + 2\left\langle \begin{matrix} 3 \\ 1 \end{matrix} \right\rangle$$

Remember that $\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle$ denotes the number of ways to order-partition $n$ people into $k$ groups.

(b) Using Problem 7, find a relationship between $\left\langle \begin{matrix} 4 \\ 3 \end{matrix} \right\rangle$, $\left\langle \begin{matrix} 3 \\ 3 \end{matrix} \right\rangle$, and $\left\langle \begin{matrix} 3 \\ 2 \end{matrix} \right\rangle$.

(c) Find a recursive rule for $< n, k >$

(d) Does your rule work in the table you made in problem 4?

**9**. (a) On a 5-button Simplex Lock, find the number of combinations that use all five buttons and contain exactly 3 pushes.

(b) On a 5-button Simplex Lock, find the number of combinations that use all five buttons.


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**10**. Let $T(m)$ be the number of combinations on an $m$-button Simplex Lock, and $L(m)$ be the number of combinations on an $m$-button Simplex Lock that use all $m$ buttons. Fill in this table!!

| $m$ | $L(m)$ | $T(m)$ |
|---|---|---|
| 1 | | |
| 2 | | |
| 3 | | |
| 4 | | 150 |
| 5 | | 1082 |

Huh?

**11**. Make a difference table for $y = x^6$, then see if you can use it to find the number of combinations on a 6-button Simplex Lock.

## Tough Stuff

**12**. Use the rules of common differences and function map sets to explain what is going on in problem 5 and how it relates to the Simplex Lock.

**13**. (a) Explain why

You're choosing 3 people out of 8. Split the 8-person set into two equal pieces.

$$\binom{8}{3} = \binom{4}{0}\binom{4}{3} + \binom{4}{1}\binom{4}{2} + \binom{4}{2}\binom{4}{1} + \binom{4}{3}\binom{4}{0}$$

(Don't just say that they're both equal to 56.)

(b) Give a "committee" proof of this identity:

$$\binom{2n}{r} = \sum_{k=0}^{r} \binom{n}{k}\binom{n}{r-k}$$

**14**. Use these problems to prove that

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$$

# 15 *Lock, Stock, and Barrel... OK, Maybe Just Lock*

**What Have We Done?!?!**

There are two sections here. The second section attacks Simplex Lock. Do what you like!

Fill in the blank with your name. Look, you're in the problem set!

**1**. ______________ is working on problem 1. This person is asked to find the number of trains of length 7 that have exactly 3 cars. How many are there?

**2**. An ice cream store claims to offer over 1,000 possible three-scoop cones of ice cream with different flavors. On further review, they're wrong, it's only 990. How many flavors of ice cream does this shop have?

**3**. Explain why $\binom{10}{3}$ is the same number as $\binom{10}{7}$.

**4**. Without a calculator, find $1001^5$.

**5**. Hal says he can find $2001^5$ almost as quickly. How would he do it?

**6**. How many different ten-letter "words" could you make with the letters CROW T ROBOT?

**7**. Kevin (The Postman), uses only 7¢ and 10¢ stamps. Which postage can Kevin *not* make: 44¢, 45¢, 46¢, or 47¢?

**8**. Table 9 flips nine coins, in honor of whatever it is they keep babbling about. Find:

(a) ... the probability that Table 9 flips exactly three heads and six tails.

(b) ... the probability that Table 9 flips at least five heads.

(c) ... the probability that Table 9 flips an *odd* number of heads.

(d) ... the connection between this problem and the polynomial $(h + t)^9$.


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**9**. Kevin (a different Kevin) takes common differences for a polynomial. He finds that the sixth common differences are all equal to 2,160.
(a) What degree is Kevin's polynomial?
(b) What is the first coefficient of Kevin's polynomial?

This problem does not require separation, or bacon.

**10**. Mr. Sicherman shows you two dice: one has sides 1, 2, 2, 3, 3, 4, and the other has sides 1, 3, 4, 5, 6, 8.
(a) Write a polynomial that represents each die.
(b) You roll each die once and add up the numbers on the dice. How many ways are there to make each sum (and what sums are possible)? How about that!

**11**. Ten people decide to form poker games. Two of the games will have four people each, and one game will have two. How many ways could this happen, if no one cares about "seats" (ordering) within games, or what game they are put in?

So a seating with $\{A, B, C, D\}$ and $\{E, F, G, H\}$ is the same as $\{E, F, G, H\}$ and $\{D, A, C, B\}$, etc.

**12**. How could you use expanding $(r + c + w)^5$ to find the number of trains of length 6 with 4 cars and 1 weld?

**13**. On the game show *Wheel of Fish*, the winning contestant throws nine coins in the air. For every coin that comes up (fish) heads, they win 10 fresh fish as a prize. Find the mean and standard deviation for the number of fish a player can expect to win from this game.

## Breaking the Lock

**14**. Yesterday, we made a list of the 13 ways to push all three buttons on a 3-button Simplex, along with the 13 ways to push some, but not all, of the buttons on a 3-button Simplex. Can you find a *mapping* between one set and the other that explains why there are just as many of each? To be convincing, you should be able to explain:
- If I've got a member of one set, does my mapping *definitely* have to be in the other set?
- How do I know that my mapping doesn't send two members of one set to the same place in the other?

Please don't be like Nancy Reagan, and just say no.

**15**. Use problem 14 to explain why the total number of combinations on a Simplex lock is exactly twice the number of combinations that use all the buttons.

**16**. Use one of the tables from problem 4 on yesterday's set to find the total number of combinations on a 5-button Simplex lock.

**17**. Use a difference table for $y = x^6$ to find the total number of combinations on a 6-button Simplex lock.

**18**. Do you think a Simplex lock with $n$ buttons could ever have more than $10^n$ combinations?

A three-wheel combination lock has $10^3$ combinations. A five-wheel combination lock has $10^5$ combinations, although Planet Druidia didn't choose the hardest one to find.

**19**. Out of the 94,586 combinations on a 7-button Simplex Lock, how many of them use fewer than 7 buttons?

**20**. How many combinations on a 7-button Simplex lock use exactly the buttons $2, 3, 4, 6, 7$?

**21**. How many combinations on a 7-button Simplex lock use exactly five buttons?

**Tough Stuff**

**22**. Prove that an even number (greater than 2) can always be written as the sum of exactly two prime numbers.

**23**. This problem follows problem 10.

(a) How are the results of problem 10 connected to the factoring of $(x + x^2 + x^3 + x^4 + x^5 + x^6)^2$?

(b) Find *three* distinct dice that, when rolled, give the same distribution as three normal dice.

**24**. Judy decides to play some craps. The probability of winning at a game of craps is exactly $\frac{244}{495}$ (which is its own Tough Stuff problem for another time).

(a) If you bet \$10 once, find the mean and standard deviation for the amount of money you will *win*.

The mean should be negative.

(b) If you bet \$10 one hundred times, find the mean and standard deviation for the amount of money you will win.

(c) If you bet \$10 one million times, find the mean and standard deviation for the amount of money you will win. It's good to be the house.


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