1 *Over and Over*

Hey, welcome to the class. We know you'll learn a lot of mathematics here—maybe some new tricks, maybe some new perspectives on things with which you're already familiar. A few things you should know about how the class is organized:

- Don't worry about answering all the questions, ever. If you're answering every question, we haven't written the problem sets correctly.
- Don't worry about getting to a certain problem number. Some participants have been known to spend the entire session working on one problem (and perhaps a few of its extensions or consequences).
- Have fun! Make sure you're spending time working on problems that interest you. Feel free to skip problems that you're already sure about. Relax and enjoy!
- Each day's problems are divided into three categories: Important Stuff, Neat Stuff, and Tough Stuff. It's a good idea to do the problems in Important Stuff first. We try to make sure that the problems in Important Stuff can be picked up regardless of how much or little work you've done on prior sets.

Okay, so let's get started. We're going to start with doing the same thing, over and over. Okay, so let's get started. We're going to start... alright, that's enough.

PROBLEM

We're going to give your table a card with two numbers on it. Start with these two numbers, then generate a sequence by the rule

the next term is the sum of the two terms that came before

1. Generate 15 terms.
2. Describe any patterns you find.
3. How big will the 100th term be? Find an approximate answer. Or an exact answer!

When you're done, exchange cards with another group and do it again. And again, and again, and... alright, that's enough.

Important Stuff.

1. Pick a number, any number—now add one, then add one again, then again, and again. . .
(a) What do you get after 100 times?
(b) What will happen in the long run?
(c) If you can, give a formula for what happens after n iterations.

Each time you add one is an *iteration*. You'll hear that term over, and over, and ... alright, that's enough.

2. Pick a number, any number—double it, then double again, then again, and again. . .
(a) What do you get after 100 times?
(b) What will happen in the long run?
(c) If you can, give a formula for what happens after n iterations.

3. Pick a number, any number—double it and add one, then double and add one again, then again, and again. . .
(a) What do you get after 100 times?
(b) What will happen in the long run?
(c) If you can, give a formula for what happens after n iterations.

4. Pick a number, any number—divide it by two and add one, then divide by two and add one again, then again, and again. . .
(a) What do you get after 100 times?
(b) What will happen in the long run?
(c) If you can, give a formula for what happens after n iterations.

5. Pick a number, any number—take its reciprocal and add one, then take the reciprocal of *that* and add one again, then again, and again...
(a) What do you get after 100 times?
(b) What will happen in the long run?
(c) If you can, give a formula for what happens after n iterations.

Neat Stuff.

This section includes a variety of problems each day, which range from practice to extension. Pick and choose problems that catch your fancy, depending on your background and focus.

Don't be surprised to see problems here repeated in later sets; that's our way of suggesting you check it out sometime.

6. Amanda came to PCMI last summer, and when she packed up, she organized your Cuisinaire rods into baggies, each containing all the rods of the same length. But she lost all of the baggies except two of them—the bags containing rods of length 1 and 2. Now how many trains can she make of length 5? 6? 8? n? (A 2-1-2 train is different from a 2-2-1 train, by the way, as far as Amanda is concerned.)

Typical!

7. Consider the sequence $1, 1, 3, 7, 17, 41, 99, \ldots$
(a) What's a possible pattern to the sequence?
(b) Approximate the 50th term without finding it.
(c) Is there an exact formula here? ("Yes.") Good luck. Please move on to the next problem now.

We have a pattern in mind, but, as you know so well, there are many ways to extend any pattern.

8. Consider an iteration rule defined on *points* and given by

$$(x, y) \mapsto (y, x + y)$$

So, for example, $(3, 7)$ ends up at $(7, 3 + 7) = (7, 10)$.
(a) Pick your favorite point other than the origin, and apply the transformation to it. Then apply the transformation to the result. Then do it again, and again, and again, plotting your points each time. Plot at least 8 points. What happens?
(b) Use a graphing calculator to find a line of best fit for your points.

What happens if you pick the origin? Boy, what a bad idea that is.

9. Pick a number, any number—take twice its reciprocal and add one, then take twice the reciprocal and add one again, then again, and again...

(a) What do you get after 100 times?
(b) What will happen in the long run?
(c) If you can, give a formula for what happens after n iterations.

Tough Stuff.

This section has some difficult problems! Try these if you're up for a challenge or already feel pretty confident about the problems in the rest of the set. Our guarantee: something challenging every day.

10. In problem 8, you looked at the the function defined on points given by

$$(x, y) \mapsto (y, x + y)$$

(a) Find a point (a, b) that end up being *scaled* by the map. That is, find a point (a, b) that is taken to (ka, kb) for some number k.
(b) Classify all such points. No, you didn't find them all, keep trying.

11. In problem 5, you picked a number, took its reciprocal and added one, then you kept doing it until it just wasn't fun anymore.
(a) Find all numbers that are *fixed* by this operation.
(b) If ρ is such a fixed point, what is the value of

$$1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{\rho}}} \ ?$$

12. Find the value of

What does the "..." mean here?

$$\sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \dots}}}}}$$

13. Find x if

$$\sqrt{x + \sqrt{x + \sqrt{x + \sqrt{x + \sqrt{x + \dots}}}}} = 15$$