10 *Drinks and a Movie*

PROBLEM

G is the graph

Find:

1. The adjacency matrix, A
2. The matrix giving the number of 3 step walks

All the paths go both ways.

Useful Stuff.

1. Find all the fixed points for the iteration $x \mapsto x^2 - 0.5$, and determine whether each is attracting, repelling, or neither.

2. Here's a rule that governs atrioventricular conduction in mammals. No, really, it does:

$$t_n = \frac{375}{t_{n-1} - 90} + 100$$

Additionally, t_n is restricted to be at least 90.

(a) Are there fixed points? Find them any way you like.

(b) Are the fixed point(s) attracting or repelling? Use a Web diagram to decide. Hey, t_n has to be greater than 90, fool.

(c) Describe what happens with initial condition $t_0 = 200$.

3. Which of these iteration rules have an *attracting* fixed point at $x = 9$?

$$\begin{array}{ccc} x \mapsto 2x-9 & x \mapsto -2x+27 & x \mapsto \frac{18}{x}+7 \\ x \mapsto 9 & x \mapsto \frac{9}{x}+8 & x \mapsto -x+18 \\ x \mapsto \frac{x^2}{9} & x \mapsto x^2-81+x & x \mapsto \frac{2}{3}x+3 \\ x \mapsto \frac{4}{3}x-3 & x \mapsto x^2-72 & x \mapsto x \end{array}$$

Break it up, break it up! Among the group, we mean.

4. Amy bought a house! She has a 15-year mortgage at a low rate of 6% APR. That means that every year, whatever she owes grows by 6%. She owes $100,000 to start. Sadly, the interest is added first, before any payment is made.

Hurray for Amy!

(a) If Amy pays $6,000 every year, explain why the rule

$$x \mapsto 1.06x - 6000$$

describes the balance between one year and the next.

(b) If Amy pays $6,000 every year, what happens?

(c) If Amy pays $7,000 every year, will she pay off the $100,000 in 15 years? If not, how much is left at the end?

(d) If Amy pays $8,000 every year. . . same questions.

(e) What of $9,000? Hey, you noticing anything? If you're not noticing anything other than "It's getting smaller," try it for $10,000.

(f) Figure out exactly how much Amy needs to pay every month. Right now, no more guessing. Really, it's possible!

We are ignoring things like monthly payments to make it nicer. For now. Also, Amy metabolizes TaB just as fast as anyone.

SLOW DOWN! This is important right here.

5. Do each of these matrix calculations, then figure out why we asked you to do it.

(a) $\begin{bmatrix} 2 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 4 \\ 7 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 12 \\ 21 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

(d) $\begin{bmatrix} 2 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} + \begin{bmatrix} 2 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

(e) $\begin{bmatrix} 2 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 7 \\ 6 \end{bmatrix}$

By this we mean we want you to look at the results of the calculations and track down some general behavior. Heck, you might even try *proving* it with a general case.

Neat Stuff.

6. Rey is the King of Math and the King of TaB. In fact, he drinks *two* TaBs every two hours, both at the same time. Chug, chug!

(a) Write an iteration rule for the amount of caffeine from TaB running through Rey's body.

(b) Suppose Rey starts caffeine-free, like Stephanie did. Make two tables comparing Rey's caffeine intake and Stephanie's. What's the relationship?

(c) Joy learns that TaB can make singing voices even better, and vows to drink *three* TaBs every two hours. What happens to her, other than having to go to Albertson's every day? Compare to Rey and Stephanie.

If Joy keeps drinking like this, she'll run up quite a TaB after a while. Thanks, we're here all week.

7. So, let's see, we've found that the iteration $x \mapsto x^2 - 0.5$ has an attracting fixed point, and the iteration $x \mapsto x^2 - 1$ does not.

(a) What about $x \mapsto x^2 + 0.5$?

(b) What about $x \mapsto x^2 + 0.25$?

(c) Somewhere between $x \mapsto x^2 - 0.5$ and $x \mapsto x^2 - 1$ is a place where stuff goes from happy-happy to not happy-happy. Where's that?

This third one was nearly "Tough Stuff", so feel free to move on. There are a couple ways to attack this, but neither is easy.

8. Describe what the matrix

$$\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

does to any point in the plane.

9. Not to be outdone, the matrix

$$\begin{bmatrix} 0 & 1 \\ -15 & 8 \end{bmatrix}$$

says "You're not so tough, $\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$. I can scale some points, too. Maybe not all of them, but some."
Find some points, other than $(0, 0)$, that scale when multiplied by this matrix. In other words, solve this system:

$$\begin{bmatrix} 0 & 1 \\ -15 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} kx \\ ky \end{bmatrix}$$

Tough Stuff.

10. Suppose

$$A = \begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix} \text{ and } P = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$$

Find:

(a) $P^{-1}AP$ (b) $(P^{-1}AP)^2$ (c) $(P^{-1}AP)^5$

(d) $P^{-1}AP$ (e) $P^{-1}A^2P$ (f) $P^{-1}A^5P$

(g) A^5 (h) A^6 (i) A^n (n a positive integer)

11. Find an iteration rule with exactly three distinct attracting fixed points. Or, prove no such rule exists.

12. Draw a picture in the complex plane to describe the fixed point dynamics of the iteration rule

$$x \mapsto 2x + \frac{1}{x}$$

13. Consider *all* the iteration rules in the form

$$x \mapsto Ax^2 + Bx + C$$

(a) Will there always be a fixed point?
(b) Find the fixed point(s) in terms of A, B, and C.
(c) Can there be more than one fixed point? Can there be more than one *attracting* fixed point?
(d) Determine when rules in this form will have *attracting* fixed points.

21. (continued) Pafnuty shows you this recursive rule:

Pafnuty? What kind of name is that?! Course, it could just be a *hint* of some sort.

$$t_n = (2cos\theta)t_{n-1} - t_{n-2}$$

Start with the initial conditions $t_0 = 1$, $t_1 = cos\theta$.
(a) Graph each t_n: what happens?
(b) Write a closed form for t_n using what you know about two-term recurrences.