13 *Uh Oh*

Is 13 unlucky? Who knows. This problem set, though? Definitely!

PROBLEM

Tomorrow is review day. Among your table, take between 10-15 minutes to craft a problem for the review. As a reminder, we've had four major topics:

- recurrence relations
- iteration
- bad humor
- matrices

Problems that connect topics are ideal here, but not required. The biggest thing we're looking to do tomorrow is show how far we've come in three weeks.

Write your question on a card, and we'll collect 'em all.

We reserve the right to change your question in order to improve its bad humor potential, or possibly some of the numbers or what-not. We won't change it fundamentally at all.

Bioinformatical Stuff.

The problems in this section refer to the iteration rule

$$x \mapsto kx(1-x)$$

Here, x represents a proportion (from 0 to 1) of the maximum possible population that could be sustained for a generation with the resources of a system, and k is a constant affecting reproduction rate. The idea here is that the current population uses some of the resources, and depending on the population this generation, the population of the next generation might be larger or smaller. The value of k must be between 0 and 4.

Borrrring. Borrrring...

The two parts of the rule $x \mapsto kx(1-x)$ represent the potential growth and decay of population from one generation to the next:

- the kx part represents reproduction, which is proportional to the population and to a known growth rate. When the growth rate increases, quicker population growth occurs.
- the $(1-x)$ part represents resource consumption, which is proportional to how close the population is to the maximum.

Man, that was long. If this were the Gong Show, that would've been gonged long ago. And now, the Unknown Conic, followed by Gene Gene the Piano-Playin' Machine.

1. Try $k = 1.5$ and initial value $x = 0.1$. Iterate. What happens?

2. The graph of $y = kx(1-x)$ is a parabola. What are the coordinates of the intercepts? The vertex? Feel free to test values of k first.

3. Knowing what you know about the vertex, why does it have to be a positive k? Why can't k be larger than 4?

Remember, it's iteration. Also remember Positive K's hit "I Got A Man"? Yeah, me neither.

4. By hand, carefully draw a Web diagram that shows the iteration for

$$x \mapsto 2x(1-x)$$

using the starting value $x_0 = 0.1$. Find the value of the fixed point, and determine whether or not it is attracting.

Build the coordinate grid to include only values between 0 and 1 on both axes.

5. Consider $k = 2.5$. Find the value of the fixed point, and determine whether or not it is attracting. Does anything different happen?

6. What happens when $k = 0.5$?

7. What happens when $k = 3.0$?

8. What happens when $k = 3.2$?

9. What happens when $k = 3.5$?

As a reminder, you can switch from plotting a Web diagram to plotting a time series under the TI-83/84's FORMAT menu, or under the "$y =$ " tool's AXES menu on the TI-89.

10. What happens when $k = 4.0$? Yeesh. As Stefani might say, this is bananas. And all we did was change k a bit.

11. (a) What happens when $k = 4.0$ and the starting point is exactly $x_0 = \frac{5+\sqrt{5}}{8}$? Use the exact value and no calculator. *Time limit:* 2 minutes.

If this didn't make sense, that's probably a good thing. Holla back!

(b) Use a decimal approximation to $x_0 = \frac{5+\sqrt{5}}{8}$. What happens?

Amazingly Fantastic Stuff.

So, I guess we don't know all there is to know about iteration. But, fortunately, at least we know all there is to know about recurrence rules, right? Uh oh.

12. Hey, check out this recurrence rule:

$$t_n = 2t_{n-1} - t_{n-2}$$

Start with $t_0 = 3$ and $t_1 = 8$. What happens? Try a different starting point. What happens then? Oh, dear.

Your *powers* are useless against the recursive dark side.

13. Hey, check out this recurrence rule:

$$t_n = 3t_{n-1} - 3t_{n-2} + t_{n-3}$$

Start with $t_0 = 1$, $t_1 = 4$, and $t_2 = 9$. What happens? Try a different interesting starting point. What in the heck?

14. Alright, here's another one:

$$t_n = 4t_{n-1} - 6t_{n-2} + 4t_{n-3} - t_{n-4}$$

Hm? Try the starting point $(-1, 0, 1, 8)$ as t_0 through t_3.

15. Find a recursive rule that could generate a fourth-degree polynomial.

And you thought we'd be able to avoid Blaise for three weeks.

16. San Dimas High School has 480 seniors, 520 juniors, 550 sophomores, and 600 freshmen. Suppose that each year, the incoming freshman class is 10% larger than the class which just graduated, and that otherwise, the size of each class is constant from start to finish. So, for example, there will be 520 seniors, 550 juniors, 600 sophomores, and 528 freshmen next year.

San Dimas High School football *rules*!

(a) Build a matrix to describe what happens year-over-year.

(b) What will the school enrollment be 10 years from now?

(c) Will there ever be a year that the senior class is the largest? Explain why or why not.

Tough Stuff.

17. The *second iterate function* is a function that takes x to the result of 2 iteration steps. For example, if the function used for iteration is $x \mapsto x^2$, then the second iterate function is $x \mapsto x^4$.

(a) Show that the second iteration function for $x \mapsto x^2 - 1$ is $x \mapsto x^4 - 2x^2$.

(b) Find *all* the fixed points for $x \mapsto x^4 - 2x^2$ using algebra, a calculator, or a computer.

(c) Show that the second iteration function for $x \mapsto 2x(1-x)$ is $x \mapsto 4x(1-x)(1-2x+2x^2)$.

(d) Find *all* the fixed points for the second iteration function for $x \mapsto 3.3x(1-x)$.

(e) Investigate some other second iteration functions, or even some third (or higher) iteration functions to find other cycles.

One way to think of this uses function notation. If $f(x)$ is the function, then $f(f(x))$ is the second iteration function. If $f(x) = x^2$, then $f(f(x)) = f(x^2) = (x^2)^2 = x^4$.

18. Find a value of k $(0 \le k \le 4)$ so that the end behavior for

$$x \mapsto kx(1-x)$$

using initial value $x_0 = 0.1$ is:

(a) a 3-term cycle.

(b) a 4-term cycle.

(c) a 5-term cycle.

(d) a 6-term cycle.

19. A number C is in the M-set if the iteration rule

$$x \mapsto x^2 + C$$

with starting point $x = 0$ does *not* go off to infinity.

(a) Find all real numbers that are in the M-set.

(b) Find all pure imaginary numbers $(x = 0 + bi)$ that are in the M-set.

(c) Find all the numbers that are in the M-set.

Put more mathematically, the orbit of $x = 0$ is bounded.

20. Pafnuty randomly walks around the edges of a cube, starting from a vertex on one corner. Find the probability that he is at the opposite corner after exactly 8 steps (where a step takes him from a vertex to any adjacent vertex of the cube).

21. (continued) Connect all the problem 21's together into a cohesive whole. Or, just talk to Ben, he loves this crazy junk.