14 So Long, and Thanks for All the Problems

Review Stuff.

Write your name in the blank. Look! You're finally in the problem set! Now quit whining and get back to work.

1. Local celebrity __________ claims that trees in the Park City and/or Cincinnati areas are being cut at a rate of 15% per year. Authorities have decided to plant 2,000 trees every year to try and counteract this effect.

(a) What is the fixed point?

(b) If there are 50,000 trees to start, what happens in the long run?

2. (a) Find a recursive rule that fits this sequence:

$$2, 3, 11, -3, 119, -387, 2351, \ldots$$

(b) Find a closed-form rule that fits the aforementioned sequence.

3. Given the square with these vertices:

$$(2, 1), (4, 2), (3, 4), (1, 3)$$

(a) Find a transformation matrix A that rotates the square so it has these vertices:

$$(-3, 1), (-4, 3), (-2, 4), (-1, 2)$$

Really, find it. It's somewhere in the room, like those Where's Waldo books. Except this time, it's Where's . . . eh, I got nothin'.

(b) Find a transformation matrix B that *undoes* this process.

(c) Calculate $A \cdot B$.

4. *Tougher:* Find the closed-form rule for the sequence defined by

$$t_n = 10t_{n-1} - 31t_{n-2} + 30t_{n-3}$$

with initial values $t_0 = 3$, $t_1 = 10$, $t_2 = 38$.

5. Here's a 4-by-4 matrix:

$$X = \begin{bmatrix} 5 & 1 & 2 & 1 \\ 1 & 5 & 2 & 1 \\ 2 & 2 & 3 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

Matrix X gives you all the paths of *length 2* for 4 points A, B, C, and D on a map. Reconstruct the original map of the connections between the four points.

A hint? A *hint*? Oh, all right. The matrix of paths of length 3 is

$$\begin{bmatrix} 4 & 12 & 7 & 3 \\ 12 & 4 & 7 & 3 \\ 7 & 7 & 5 & 4 \\ 3 & 3 & 4 & 3 \end{bmatrix}$$

6. Every day, Art decides he's going to spend 5% less time in the classroom than the day before. However, he also ends up missing another 3 minutes on top of that for laptop-based reasons, perhaps. On the first day, he spends the entire 120 minutes in class.

(a) Determine using these rules how much time he spends in class on the second day. (111 minutes is the correct answer.)

(b) So, does Art show up at all on Day 14?

We didn't get a problem submitted from Table 6, so we made up this one. Happy trails, RAT TAT BOMB.

7. Consider the recurrence rule

$$t_n = 3t_{n-1} - 4t_{n-2}$$

(a) Find a 2-by-2 *matrix* that could be used to model this recurrence.

(b) Find two very different pairs of starting values of t_0 and t_1 that make t_n a geometric sequence.

8. In Park City, teachers demand problems at all hours of the night to avoid boredom, Algebra Al, Bioinformatics Bowen, and Bifurcation Ben all live in different condos and create problems for everyone. Arithmetic Art delivers problems to teachers in the three condos. Here's a map showing how likely Art is to go from one location to another:

Boredom can also be alleviated by Dance Dance Revolution, kicking graduate students out of places they don't belong, and by trying to decide how much is 3.2% by weight.

(a) Write a 3-by-3 matrix of probabilities.
(b) Square this matrix. What does it mean?
(c) After a whole night of deliveries, what happens?

9. In Philadelphia, the city of brotherly cheesesteak, trains leave on time. Hah. Tom wants to know how many trains he could build using only cars of lengths 1 and 4.

(a) Find the number of trains Tom can build of lengths 1 through 10.
(b) Find a recursive rule that can generate more trains of longer lengths.
(c) *Tougher:* Tom's buddy Damon Hunting says he can solve the problem with matrices. Find a matrix that could be used to make Tom's calculations quick.

W.C. Fields was once known for saying he wanted on his epitaph, "I'd rather be in Philadelphia." Fun fact of the day!

10. Theresa drinks too much TaB and falls into a saccharin-induced coma. She dreams…of becoming a millionaire.

A teacher's starting salary is \$30,000. In her dreams, salaries rise by 7% each year, and each teacher receives an extra \$1500 in Survival Bonus Pay at the end of each year (which rolls into the salary).

(a) Theresa retires after 30 years. What is her salary then?
(b) How long will it take Theresa's salary to reach \$1 million per year?
(c) *Tougher:* How many years does it take for Theresa to earn a *total* of \$1 million?

11. Here's a map showing how people move to and from tables in the morning session:

How many paths are there to Table 4 if you don't start at Table 4? Good thing, that.

Starting at Table 2, how many paths of *length 2* are there to get to Table 4? Table 11? the Food Tent?

12. A figure in the coordinate plane is defined by this 2-by-4 piece of wood, uh, matrix:

$$\begin{bmatrix} 0 & 4 & 6 & 1 \\ 0 & 1 & 3 & 5 \end{bmatrix}$$

Find *more than one* matrix that will transform this shape into one of area 66.

13. See the extra handout sheet. Nice map, Cincinnati!

If 4 Texas teachers spend 4 hours a day playing double-4 dominos, how long will it take them to make 4 times their score?

Barely Useful Stuff.

14. For each mapping problem, make a probability matrix instead of an adjacency matrix. Then, determine which of the starting points would have the greatest "PageRank" if such a thing could be done.

15. Consider the iteration rule

$$t_n = t_{n-1} + t_{n-2}$$

with $t_0 = 5$ and $t_1 = 12$. Find the first pair of consecutive terms that have a common factor greater than 1.

16. Can the sum of squares

$$1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2$$

ever be a square number (when $n > 1$)? If so, find all possible values of n.

Tough Stuff.

17. The iteration rule $x \mapsto x^2 + C$ starting from $x = 0$ sometimes goes off to infinity, and sometimes it don't.

(a) If C is a real number, find all possible values of C that do *not* send the rule $x \mapsto x^2 + C$ off to infinity.

(b) If C is a pure imaginary number $(0+bi)$, do the same.

(c) All the numbers C that "work" under this rule form a set that covers part of the complex plane. How big is that?

18. So, about this $x \mapsto x^2 + C$.

(a) For what values of C will this have an attracting fixed point? (We're back to real numbers now.)

(b) What happens as C gets smaller? Test some values.

The phrase "period-doubling bifurcation" comes to mind. Really, I hope that phrase never comes to your mind.

19. Suppose n is a positive integer. Is there a right triangle with rational numbers as its side lengths with area n?