2 *. . . and Over Again*

PROBLEM

We're going to give your table a card with pairs of numbers on it. Start with each pair of two numbers, then generate a sequence by the rule

the next term is double the one before,
plus three times the one before that

1. Generate 10 terms for each pair.
2. Describe any patterns you find, either within a sequence or between sequences.
3. Find a closed-form rule for as many of the pairs as possible.

What? Didn't we do this yesterday, and now we're doing it over again?

So if the first two terms were 3 and 7, the next would be 23.

Important Stuff.

1. Pick a number, any number—take its reciprocal and add one, then take the reciprocal of *that* and add one again, then again, and again. . .
   (a) Calculate the first 15 terms, going from $n = 0$ to $n = 14$. Fraction? Decimal? You decide! It might matter!
   (b) What will happen in the long run?
   (c) Describe how someone could find the term when $n = 100$ without having to find the term when $n = 99$.

Please don't do this problem over again if you already did.

2. Pick a number, any number—take its reciprocal, multiply by three, then add two. Lather, rinse, repeat.

(a) Calculate the first 10 terms, going from $n = 0$ to $n = 9$. Fraction? Decimal? You decide! It might matter!
(b) What will happen in the long run?

3. Use a graphing calculator to sketch the graph of

$$y = \frac{3}{x} + 2$$

Are there any points on this graph where $y = x$? Okay, what are they?

4. Consider the sequence defined by

$$t_n = \begin{cases} 2 & \text{if } n = 0 \\ 1 + t_{n-1} & \text{if } n > 0 \end{cases}$$

Read this as "The sequence starts with 2 and each term after that is one more than the term before it."

(a) Find $t_1, t_2, \ldots, t_6$.
(b) What is a formula for t_n in terms of n?
(c) Find $t_1 + t_2 + \ldots + t_6$.
(d) What is the sum of the terms up to and including t_{12}?

5. Consider the sequence defined by

$$t_n = \begin{cases} 1 & \text{if } n = 0 \\ 2 \cdot t_{n-1} & \text{if } n > 0 \end{cases}$$

Read this as "The sequence starts with 1 and each term after that is twice the one before it."

(a) Find $t_1, t_2, \ldots, t_6$.
(b) What is a formula for t_n in terms of n?
(c) Find $t_1 + t_2 + \ldots + t_6$.
(d) What is the sum of the terms up to and including t_{12}?

6. Consider an iteration rule defined on *points* and given by

$$(x, y) \mapsto (y, x + y)$$

So, for example, $(3, 7)$ ends up at $(7, 3 + 7) = (7, 10)$.
(a) Pick your favorite point other than the origin, and apply the transformation to it. Then apply the transformation to the result. Then do it again, and again, and again, plotting your points each time. Plot at least 8 points. What happens?

What happens if you pick the origin? Boy, what a bad idea that is.

(b) Use a graphing calculator to find a line of best fit for your points.

7. Consider the sequence defined by

$$t_n = \begin{cases} 6 & \text{if } n = 0 \\ 2 \cdot t_{n-1} + 1 & \text{if } n > 0 \end{cases}$$

(a) Find $t_1, t_2, \ldots, t_6$.
(b) What is a formula for t_n in terms of n?

8. Consider the sequence defined by

$$t_n = \begin{cases} \text{-1} & \text{if } n = 0 \\ 2 \cdot t_{n-1} + 1 & \text{if } n > 0 \end{cases}$$

(a) Find $t_1, t_2, \ldots, t_6$.
(b) What is a formula for t_n in terms of n?

Neat Stuff.

9. Consider the sequence defined by

Read this as "The sequence starts with 2 and each term after that is 2 more than the one before it."

$$t_n = \begin{cases} 2 & \text{if } n = 0 \\ 2 + t_{n-1} & \text{if } n > 0 \end{cases}$$

(a) Find $t_1, t_2, \ldots, t_6$.
(b) What is a formula for t_n in terms of n?
(c) Find $t_1 + t_2 + \ldots + t_6$.
(d) What is the sum of the terms up to and including t_{12}?

10. Amanda came to PCMI last summer, and when she packed up, she organized your Cuisinaire rods into baggies, each containing all the rods of the same length. But she lost all of the baggies except two of them—the bags containing rods of length 1 and 2. Now how many trains can she make of length 5? 6? 8? n? (A 2-1-2 train is different from a 2-2-1 train, by the way, as far as Amanda is concerned.)

It's ROD TIME!

11. When Rani left PCMI, she lost some of her baggies of Cuisinaire rods, too—but she only lost the bag containing the rods of length 1. How many trains can she make of length 5? 6? 8? n?

12. Consider the sequence $1, 1, 3, 7, 17, 41, 99, \ldots$
(a) What's a possible pattern to the sequence?
(b) Approximate the 50th term without finding it.
(c) Is there an exact formula here? ("Yes.") Good luck. Please move on to the next problem now.

We have a pattern in mind, but, as you know so well, there are many ways to extend any pattern.

13. An 8×8 square can be dissected into a 5×13 rectangle:

Wow, how about that!

(a) Try it! It's fun.
(b) What's going on?
(c) Make up a puzzle that starts "A a 21×21 square can be cut into a 13×34 rectangle ..."
(d) If F_n is the nth Fibonacci number, what's the relationship between F_n^2 (the square of F_n) and the product $F_{n-1} \cdot F_{n+1}$?

Tough Stuff.

14. A sequence is defined as

$$t_n = \begin{cases} S & \text{if } n = 0 \\ T & \text{if } n = 1 \\ -t_{n-2} & \text{if } n > 1 \end{cases}$$

Find nonzero values of S and T so that t_n is a geometric sequence.

15. A sequence is defined as

$$t_n = \begin{cases} S & \text{if } n = 0 \\ T & \text{if } n = 1 \\ t_{n-1} - t_{n-2} & \text{if } n > 1 \end{cases}$$

Find nonzero values of S and T so that t_n is a geometric sequence.

16. Find all numbers x so that

$$x^n = 10x^{n-1} - 21x^{n-2}$$

for all $n > 1$.