3 ...and Over Again

PROBLEM

We're going to give your table a card with pairs of numbers on it. Start with each pair of two numbers, then generate a sequence by the rule

the next term is seven times the one before,
minus ten times the one before that

1. Generate 10 terms for each pair.
2. Describe any patterns you find, either within a sequence or between sequences.
3. Find a closed-form rule for as many of the pairs as possible.

What? Didn't we do this yesterday, and now we're doing it over again?

So if the first two terms were 3 and 7, the next would be $7 \cdot 7 - 10 \cdot 3 = 19$.

Useful Stuff.

1. Here's another iteration rule defined on points:

$$(x, y) \mapsto (y, 3x + 2y)$$

So, for example, $(3, 7)$ ends up at $(7, 3 \cdot 3 + 2 \cdot 7) = (7, 23)$.

(a) Pick your favorite point other than the origin, and apply the transformation to it. Then apply the transformation to the result. Then do it again, and again, and again, plotting your points each time. Plot at least 8 points. What happens?

(b) Use a graphing calculator to find a line of best fit for your points.

What happens if you pick the origin? Boy, what a bad idea that is.

2. The sequence $1, 2, 11, 43, 184\ldots$ comes from the recursive rule

$$t_n = At_{n-1} + Bt_{n-2}$$

(a) Find A and B.
(b) Find the next term.
(c) See Tough Stuff for a harder problem.

So it's A times the one before, and B times the one before that, and we want you to figure out what A and B are.

3. Consider the sequence defined by

$$t_n = \begin{cases} 3 & \text{if } n = 0 \\ t_{n-1} + 5 & \text{if } n > 0 \end{cases}$$

(a) Find $t_1, t_2, \ldots, t_6$.
(b) What is a formula for t_n in terms of n?
(c) Find $t_0 + t_1 + t_2 + \ldots + t_6$.
(d) What is the sum of t_0 through t_{12}?

Read this as "The sequence starts with 3 then a bunch of stuff happens."

4. Consider the sequence defined by

$$t_n = \begin{cases} 1 & \text{if } n = 0 \\ 3 \cdot t_{n-1} & \text{if } n > 0 \end{cases}$$

(a) Find $t_1, t_2, \ldots, t_6$.
(b) What is a formula for t_n in terms of n?
(c) Find $\sum_{k=0}^{6} t_k$. What the heck does this mean? Oh, it's the same question as 3c.
(d) What is the sum of t_0 through t_{12}?

Read this as "The sequence starts with 1 and each term after that is triple the one before it."

5. Amanda came to PCMI last summer, and when she packed up, she organized your Cuisenaire rods into baggies, each containing all the rods of the same length. But wouldn't you know it, only the bags of whites (length 1) and reds (length 2) showed up.

(a) How many trains can Amanda make of length 5? 6? 7? (A 2-1-2 train is different from a 2-2-1 train.)
(b) Describe a pattern, using a difference equation. Explain your pattern to someone else at your table who thinks Excel is better than Cuisenaire (it ain't).

It's ROD TIME! Yes, get out there, get the rods, play with the rods. Or, we promise to make fun of you directly in tomorrow's set.

6. Rani had some rods too, but she lost the bag of whites. Now she's only got rods of length 2 and longer.

(a) How many trains can Rani make of length 5? 6? 7? (A 4-3 train is different from a 3-4 train.)
(b) Describe a pattern, using a difference equation.

Neat Stuff.

7. In the "algebra of sequences" we've been using, the rules allow you to say things like

$$3 \cdot (1, 5) + 4 \cdot (2, 1) = (3, 15) + (8, 4) = (11, 19)$$

(a) Find A and B so that

$$A \cdot (1, 5) + B \cdot (-1, -2) = (10, 17)$$

(b) A sequence starts with $10, 17, \ldots$ and follows the rule

$$t_n = 7t_{n-1} - 10t_{n-2}$$

Find the next 6 terms.

(c) Use the result of problem 7a and the card stuff to find a closed-form rule for the sequence in problem 7b.

Nicole says that picking zero would be a bad idea. How does this make you feel? Write a paragraph or two. Hey, why not start with the number $\frac{17}{10}$. Oh, no reason.

8. Pick a number, any number—take its reciprocal, multiply by -10, then add seven. Keep on truckin'. Ack, what horrible numbers. Or are they. . .

(a) Calculate the first 10 terms, going from $n = 0$ to $n = 9$, as fractions.

(b) What will happen in the long run?

(c) Find all the numbers that solve the equation

$$x = \frac{-10}{x} + 7$$

9. Here's another iteration rule defined on points:

$$(x, y) \mapsto (y, -x)$$

So, for example, $(3, 7)$ ends up at $(7, -3)$.

(a) Pick your favorite point other than the origin, and apply the transformation to it. Then apply the transformation to the result. Then do it again, and again, and again, plotting your points each time. Plot at least 8 points. What happens?

(b) What about the iteration rule

$$(x, y) \mapsto (x, -y)$$

10. Consider the iteration rule

$$x \mapsto 3 + \frac{5}{x}$$

Good, now you've considered it.

(a) Pick any *NONZERO* number x to start with, and calculate the first 9 terms of the iteration. Express your answers as fractions.

(b) The numerators of these fractions form a sequence. Find a recursive rule that describes how to get one numerator from previous numerators:

$$N_n = \cdots$$

(c) Hey, let's not leave the denominators hanging. Find another recursive rule that describes how to get one denominator from previous denominators:

$$D_n = \cdots$$

Tough Stuff.

11. Here's another iteration rule defined on points:

$$(x, y) \mapsto \left(\frac{\sqrt{3}}{2}x - \frac{1}{2}y, \frac{1}{2}x + \frac{\sqrt{3}}{2}y\right)$$

So, for example... aw, it's Tough Stuff, you don't get an example.

(a) Pick your favorite point other than the origin, and apply the transformation to it, using exact values. Keep doing this until you see what is happening.

(b) Find another rule that does something similar with a different number of points.

12. Write a closed form for the sequence in problem 2.

13. A sequence is defined as

$$t_n = \begin{cases} S & \text{if } n = 0 \\ T & \text{if } n = 1 \\ -t_{n-2} & \text{if } n > 1 \end{cases}$$

Find nonzero values of S and T so that t_n is a geometric sequence.