4 . . . and Ova Reagan

PROBLEM

Start with the pair of numbers 7 and 10, and generate a sequence by the rule

the next term is zero times the one before,
minus zero times the one before that

1. Generate 10 terms of this sequence.
2. Describe any patterns you find.

So if the first two terms were 3 and 7, the next would be $0 \cdot 7 - 0 \cdot 3 = 0$.

Useful Stuff.

Okay, enough of those problems in boxes. Now, solve these problems with boxes. . . of RODS!

Listen for the collective cheer *and* groan from the crowd at this. Very rare, this. Aki says we shouldn't force people to do anything, but we're forcing you to put down your Excel and do this! Leave the dark side. . .

1. Jeff (from Ohio) only likes to make trains with rods of length 1 and 2: the white rods and the red rods.
 (a) Find the number of trains Jeff could build of each length from 1 to 10.
 (b) These numbers seem to be heading toward some common ratio (geometric-like). What's the common ratio?

2. Jeff (from Ohio) only likes to make trains with rods of length 1 and 3: the white rods and the. . . uh. . . rods of length 3.
 (a) Find the number of trains Jeff could build of each length from 1 to 10.

(b) Write a recursive rule (also called a difference equation) describing the relationship between outputs here.

3. Meghan likes coloring, so she decides that the whites should be colored…sometimes. So she wants to do problem 1 with whites, reds, and green-colored rods of length 1 (which she happily provides to everyone, even Cincinnati).

If Natan carried these rods around to everyone, would that make him a Rod Steward?

(a) Find the number of trains Meghan could build of each length from 1 to 10. To help you get started, there are 2 possible trains of length 1 now, and a total of 5 possible trains of length 2.

(b) Write a recursive rule describing the relationship between outputs here.

(c) See "Tough Stuff" for more.

4. Megan also likes coloring, so she decides that the reds should be colored differently, sometimes. She also dislikes the number 1 due to a childhood incident, so she only uses her two types of rods of length 2 (red and pink) and the rods of length 3.

Please don't ask her about it, we're not really sure what happened either.

(a) Find the number of trains Megan could build of each length from 1 to 10. To help you get started, there are 0 possible trains of length 1 now, and 2 possible trains of length 2.

(b) Write a recursive rule, yeah, you've seen this question before.

(c) Say, are these numbers heading toward a common ratio? What's the ratio if there is one? Can you describe a rule that could take you directly to the answer for length-15 trains?

Neat Stuff.

Okay, so we're going to give you a bunch of recursive rules, and a super-convenient starting point. For each one, your job is to find the closed-form rule for the sequence.

5. $t_n = 5t_{n-1} - 6t_{n-2}$. Starting point: $(2, 5)$

6. $t_n = 3t_{n-1} + 10t_{n-2}$. Starting point: $(2, 3)$

7. $t_n = 13t_{n-1} - 30t_{n-2}$. Starting point: $(2, 13)$

8. $t_n = -7t_{n-1} - 12t_{n-2}$. Starting point: $(2, -7)$

9. $t_n = 7t_{n-1} - 12t_{n-2}$. Starting point: $(2, 7)$

10. $t_n = 6t_{n-1} + 40t_{n-2}$. Starting point: $(2, 6)$

11. So what's going on here? What relationship is there between the numbers in the recursive rule and the closed form?

12. Well what about this one, with starting point $(1, 2)$?

$$t_n = 3t_{n-1} + 5t_{n-2}$$

Does something different happen? Is there anything that can be done about this travesty?

Gee, I feel like we might have seen this sequence on Day 3. I might be wrong. Nah.

13. Find the sum of each of these:
(a) $1 + 4$
(b) $1 + 4 + 16$
(c) $1 + 4 + 16 + 64$
(d) $1 + 4 + 16 + 64 + 256$
(e) $\sum_{k=0}^{5} 4^k$
(f) Describe the general rule, which would equal

$$\sum_{k=0}^{n} 4^k$$

This weird notation is just the next in this chain.

14. Find the sum of each of these:
(a) $4 + 16$
(b) $4 + 16 + 64$
(c) $4 + 16 + 64 + 256$
(d) $\sum_{k=0}^{4} 4^{k+1}$
(e) Describe the general rule, which would equal

$$\sum_{k=0}^{n} 4^{k+1}$$

Just the next in the chain again.

15. What's the sum of the first 100 odd numbers? (The first "odd number" is 1 for the purposes of this problem.)

16. Here's another iteration rule defined on points:

$$(x, y) \mapsto (-y, x)$$

So, for example, $(3, 7)$ ends up at $(-7, 3)$.
(a) Pick your favorite point other than the origin, and apply the transformation to it. Then apply the transformation to the result. Then do it again, and again, and again, plotting your points each time. Plot at least 8 points. What happens?

(b) What about the iteration rule

$$(x, y) \mapsto (x, -y)$$

Tough Stuff.

17. The numbers in problem 3 sure look like they could use a closed-form rule. Why don't you go find it?

18. The numbers in problem 2 sure look like they're headed toward a common ratio. What is it? No, I wanted the exact answer.

19. You make all 512 trains of length 10, with no restriction on rod lengths used (no special colors, either). How many whites did you use?

20. Here's a recursive rule:

$$t_n = 19t_{n-2} - 30t_{n-3}$$

Here, we'll even give you three starting numbers: 8, 3, 79. Find the next two terms (oh wait, it's some negative number and then some four-digit number with a bunch of ones). Okay, find the closed-form rule.

21. Use algebra to prove each of these identities.

(a)

$$\alpha^n + \beta^n = (\alpha + \beta)\left(\alpha^{n-1} + \beta^{n-1}\right) - \alpha\beta\left(\alpha^{n-2} + \beta^{n-2}\right)$$

(b)

$$c\alpha^n + d\beta^n = (\alpha + \beta)\left(c\alpha^{n-1} + d\beta^{n-1}\right) - \alpha\beta\left(c\alpha^{n-2} + d\beta^{n-2}\right)$$

(c) What use is this?!

PROBLEM

Why are you reading this? What, just because it's in a box, it's suddenly important? Nothing to see here, move along.