8 *Low-TeX Thursday?*

PROBLEM

John credits his teaching success to drinking lots of TaB. Each can of TaB has 46.8 mg of caffeine.

John (starting caffeine-free) drinks a can of TaB to start (time $t = 0$) and every two hours after that. Suppose the body metabolizes $\frac{1}{4}$ of the caffeine in the bloodstream during each 2-hour period (really, that's pretty close). How much caffeine will John have in his body after 6 hours? 12 hours? in the long run? Assume John wakes up in the middle of the night to drink TaB too.

Instead, what would happen if John started with 500 mg in the bloodstream, then continued drinking TaB every two hours?

Come on, don't tell us you've never even seen a can of TaB. It's the uncola! No, it's not. It's refreshingly crisp! No, wait, that's Fresca. Anyway. John also says to spend no more than 5 minutes on this problem; if you think you need more time, move on and come back later.

Useful Stuff.

1. Consider the iteration rule

$$x \mapsto 0.75x + 46.8$$

(a) Find all numbers x where the input equals the output.
(b) Suppose you start with $x = 500$ and iterate. What happens?

2. Consider the iteration rule

$$x \mapsto \frac{6}{x} + 1$$

(a) Find all numbers x where the input equals the output. These are called *fixed points* for the iteration.
(b) Suppose you start with $x = 3.01$ and iterate. What happens?
(c) Suppose you start with $x = -2.01$ and iterate. What happens?
(d) Suppose you start with $x = -1.99$ and iterate. What happens?

3. Here's a recursive rule:

$$t_n = t_{n-1} + 6t_{n-2}$$

What happens to the ratio between terms $\frac{t_n}{t_{n-1}}$ for each of these starting points?
(a) $(2, 1)$
(b) $(2, 5)$
(c) $(2, 6)$
(d) $(2, -4)$
(e) $(2, -3.999)$

There's a reason we give 5 different ones here. We want you to try them all. Or, divide the work among tablemates. Excelleration will be tolerated, barely.

4. Find two functions where 9 is a fixed point.

5. A fixed point is called *attracting* if numbers near the fixed point move closer to the fixed point.
Which of these iteration rules have attracting fixed points?
(a) $x \mapsto 0.75x + 46.8$
(b) $x \mapsto 0.75x - 12$
(c) $x \mapsto 1.1x + 2$
(d) $x \mapsto 6$
(e) $x \mapsto -0.9x + 10$
(f) $x \mapsto -1.3x - 12$
(g) $x \mapsto x + 4$
(h) $x \mapsto -x + 4$

There is a more technical definition involving neighborhoods and the letter ϵ, but... mep.

6. When will the iteration rule $x \mapsto Ax+B$ have an attracting fixed point? Does it depend on A, B, or both?

7. Here's a rule defined on points:

$$(x, y) \mapsto (2x + 3y, 5x - 7y)$$

Find the point (x, y) that outputs the point $(11, 13)$.

8. Solve this system of equations by any means necessary:

$$2x + 3y = 11$$

$$5x - 7y = 13$$

9. Solve this system of equations:

$$\begin{bmatrix} 2 & 3 \\ 5 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11 \\ 13 \end{bmatrix}$$

Yes, it really is a system of equations. Perhaps you've seen it recently? A little *too* recently, maybe?

Neat Stuff.

Skip problems 10 through 12 until we've done our L-shaped demonstration.

10. Use graph paper to determine the effects of each of these matrices on our happy little L:

(a) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & 0.5 \\ -0.5 & 0 \end{bmatrix}$

(e) $\begin{bmatrix} 2.5 & 0 \\ 0 & 0.4 \end{bmatrix}$

11. According to our discussion, how much time should you be spending on this problem?

12. The area of the original L shape is 11 square somethings. Find the area of the new L under each of the matrices in problem 10.

13. Find the two fixed points for this iteration rule:

$$x \mapsto 2x^2 - 3$$

What happens if you try iterating numbers that are near each fixed point?

14. Find the two fixed points for this iteration rule:

$$x \mapsto \frac{x + \frac{5}{x}}{2}$$

(a) What happens if you try iterating numbers that are near each fixed point?
(b) What might this iteration rule be used to do?

15. Consider this iteration rule:

$$x \mapsto x^2$$

(a) What are the fixed points?
(b) Are any of the fixed points attracting?

16. Consider this iteration rule:

$$x \mapsto x^3$$

(a) What are the fixed points?
(b) Are any of the fixed points attracting?

17. Explain why the iteration rule

$$x \mapsto p(x)$$

where $p(x)$ is a polynomial of degree n, can't have more than n fixed points.

We can't think of a funny comment here, so consider this as though it were some kind of witty joke about Jar Jar Binks or perhaps just Steve.

Tough Stuff.

18. Bill says that any iteration rule

$$x \mapsto p(x)$$

where $p(x)$ is a cubic polynomial, *must* have a fixed point. Is he right? Prove it!

19. Find an iteration rule with two attracting fixed points at $x = 2$ and $x = 8$.

20. Find an iteration rule with exactly *three* attracting fixed points.

21. Experiment with this recursion:

$$t_n = 2x \cdot t_{n-1} - t_{n-2}$$

where the starting point is $t_0 = 1$, $t_1 = x$. Polynomials, hey-o!