# 12 *My Own Private PCMI*

**Important Stuff.**

What? The Important Stuff is first? Guess you better do that first, then!

**1**. Let $z$ be a complex number with magnitude 3. Plot a circle that contains $z$, allowing for the radius to be changed. If set $S$ is this circle, find the output when $S$ is the input to the rule

$$z \mapsto z^2 - z$$

**2**. *Use the sketch from the last problem* to find one solution to each equation.

(a) $z^2 - z = 6$

(b) $z^2 - z = 12$

(c) $z^2 - z = -9$ (approximate quick)

(d) $z^2 - z = 20$

As Roy Scheider might say, "We're going to need a bigger circle."

**3**. Let $S$ be a circle with magnitude 2. What's the output look like when $S$ is the input to the rule

$$z \mapsto z^3 + z + 2$$

**4**. Adapt the sketch in the last problem to find or approximate the *three different* solutions to the equation

$$z^3 + z + 2 = 0$$

**5**. Consider the rule

$$z \mapsto z^2 + 4z + 5$$

(a) Susan uses a really small circle $S$ as input. What does the output look like? By small we mean small.

(b) Ralph uses a really large circle $S$ as input, and grabs Susan's circle, dragging it until it's huge. What happens? Does the output ever cross the origin? How many times?

Yes, yesterday was actually Free Slurpee Day. We hope you believed us.



**Sketch of the Day**

Go back and do the Important Stuff. Oh you're done? Fine then, find (approximately) the five complex numbers that make $z^5 = 1$. Don't do this by factoring, it's the Sketch of the Day! While you're at it, find the five complex numbers that make $z^5 = -1$. That's all.

**Not So Critically Important Stuff.**

**6.** (a) Plot the two solutions to $x^2+6x = -25$, using Sketchpad or otherwise.
(b) Plot the two solutions to $4x^2 + 12x = -25$.
(c) Plot the two solutions to $9x^2 + 18x = -25$.
(d) What's going on, is there a pattern here?

**7.** Prove that any quadratic equation with real coefficients must have exactly two roots in the complex plane, except when those roots are the same point.

**8.** Consider the function $f(x) = x^3 + 6x + 3$. Don't worry, we're not going to ask you anything about the complex plane in this problem.
(a) Find a number $a$ so that $f(a) > 0$.
(b) Find a number $b$ so that $f(b) < 0$.
(c) Explain how you know that there *must* be a real number $c$ so that $f(c) = 0$.

**9.** Suppose a function $g(x)$ has $g(a) > 0$ and $g(b) < 0$. Must there always be a $c$ so that $g(c) = 0$? Explain or provide examples.

**Neat Stuff.**

**10.** Do some division! Build a triangle in the complex plane, then a point outside the triangle. Using locus constructions, divide the point into the triangle. You're going to need 3 locuses, loci, loca, whatever, one for each segment of the triangle. What does the resulting shape look like?

So to do the division, click on the isolated point first, then the point on the edge of the triangle.

**11.** What changes when the origin is inside the triangle? Why?

**12.** What changes when the origin is *on* the triangle? Why?

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**13**. What about angles in that last construction? How would angles even get measured in these absurd new shapes?

There should be plenty of Neat and/or Tough Stuff to go around from Set 11, too. Feel free to try any group of these problems. Although many are related to the core goals of the course, most are side quests to try whenever. Enjoy!

**14**. The function $f(z) = z^3 + 4z$ is an *odd* function. What would an odd function look like under these complex mappings we've been doing?

**15**. While you're at it, what about even functions?

**16**. Use the results from the Sketch of the Day to find the ten numbers that make $z^{10} = 1$.

**17**. Let $f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$. The domain of $f(x)$ is real numbers with $-1 \le x \le 1$.

Look out, it's a Taylor series! RUN! Oh, sorry, it's only a Maclaurin series. False alarm.

(a) Approximate $f(\frac{1}{5})$ to six decimal places. You might need to take a few terms.

(b) Approximate $f(\frac{1}{239})$ to six decimal places. More terms, or less terms?

(c) Evaluate $16f(\frac{1}{5}) - 4f(\frac{1}{239})$ to five decimal places.

(d) Hey, we forgot: find the direction of the complex number

$$\frac{(5+i)^4}{239+i}$$

Wacky.

**18**. Consider this set of polynomials: $P_0(x) = 1$, $P_1(x) = x$ to start. From then on, polynomials are generated by the rule

$$P_n(x) = 2x \cdot P_{n-1}(x) - P_{n-2}(x)$$

Investigate these polynomials: what is their behavior? What roots do they have?

**19**. Last night's Pizza & Problem Solving offered a question about numbers like 101, 10101, 1010101, etc. Try squaring these numbers and see what happens. Can you prove it?

**Tough Stuff.**

**20**. Turns out that $x^5 - 1 = (x-1)(x^4 + x^3 + x^2 + x + 1)$. That's not tough. The tough part is to factor that quartic piece into two quadratic factors, where each quadratic factor might have real coefficients (not just integers or fractions). Does this give you any insight into how to *construct* a regular pentagon?