Is that the Fundamental Theorem of Algebra? Man oh man.

13 *Sweet Home PCMI!*

Sketch of the Day

Investigate the behavior of the function $f(z) = z^2 - 2z + 4$ as it operates on different "magnitude circles" (circles centered at 0). Estimate with very poor accuracy the two solutions to $z^2 - 2z + 4 = 0$. (They are both on the same magnitude circle.)

Investigate the same behavior for the function $f(z) = z^2 - 4z + 4$.

Today's problem set is inspired by Lynyrd, not Reese, and is brought to you by the letter i, and the number i.

Not to worry, the plague of locus is almost complete.

Important Stuff.

1. What two numbers have a sum of 2 and a product of 4?

2. Consider the function $f(x) = x^3 + 6x + 3$. Estimate $f(\frac{1}{100})$ and $f(100)$.

3. Consider the function $f(z) = z^3 + 6z + 3$.
(a) What does the output of $f(z)$ look like when you use a very small magnitude circle?
(b) What does the output of $f(z)$ look like when you use a very large magnitude circle?
(c) Explain how you know that there *must* be a magnitude circle that contains a point z with $f(z) = 0$.

Say, this is the same function we used in the last problem!
If you can't tell, your circle isn't small enough yet.

4. Let $f(x) = x^3 + 12x - 4$. Give a good, quick estimate for each of these. Emphasis on "quick".
(a) $f(\frac{1}{100})$
(b) $f(-\frac{1}{100})$
(c) $f(100)$
(d) $f(-100)$

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5. Shenaz considers the function $f(z) = z^3 + 12z - 4$. *Without using Sketchpad*, she and the rest of you shall answer the questions of problem 3 about this other function.

6. Convince yourself, Rebecca, and everyone else that this is true:

> Let $f(z)$ be a polynomial of degree n. Then there must be at least one solution to the equation $f(z) = 0$ in the complex plane.

"Degree n" just means that the polynomial starts $f(z) = az^n + \cdots$

Neat Stuff.

7. Consider the function $f(z) = z^2 - z$.
(a) Find a solution to $z^2 - z = 1$ using Sketchpad.
(b) Find a way to use this sketch to build a golden rectangle. Is there more than one length in this diagram that could produce a golden rectangle?

8. Use Sketchpad to approximate the five solutions to $x^5 + 2x^2 + 10 = 0$. How many of the roots are real numbers?

9. Use Sketchpad to approximate the four solutions to $x^4 + 3x^2 + 1 = 0$. Yes, there are four solutions and not two. What happens, in general, to even functions? (Tougher: Find the roots by factoring.)

10. Track down the solutions to $x^3 - 7x^2 + 15x - 9 = 0$. What does a "double root" look like in the complex plane? How many solutions does this equation have?

11. Once you calculate the roots of a given polynomial, try plotting them as points in the complex plane, then seeing how the function behaves as it wanders by them. Or you could try working with circles that aren't centered at the origin. Check out

What's a "winding number"?

http://www.keypress.com/sketchpad/general_resources/recent_talks/complex_ictmt6/downloads/complex_functions.pdf

12. **Continuity!** On Saturday Donald woke up at 8 am, climbed to the top of Mt. Timpanogos, then camped out there overnight. On Sunday, Donald woke up at 8 am, then climbed down the same trail he climbed up. He didn't climb at any consistent speed, and stopped to take in the views many times. Was there definitely some time during

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the day that Donald was in the same spot (on the trail) at the same time on both Saturday and Sunday?

13. The current record for calculating π is into the trillions of digits. One completely wacky formula for calculating π comes from this calculation:

$$\frac{(N+i)^{12} \cdot (57+i)^{32} \cdot (110443+i)^{12}}{(239+i)^5}$$

Oh wait, we forgot N. Find the integer N so that this entire product has direction exactly 45°. This formula was discovered by a high school teacher in 1982. Perhaps you could find some more?

Review Your Stuff.

Historically the final day is considered review. Because this is a self-reflective process of discovery, we think that an end product of this discovery might be some summarizing questions of what you might find valuable in this course. We would like these to get at what you think are important mathematical themes in our course, and also themes that might apply to what you teach in your own school. We hope this will be a valuable journey, but mostly we're just lazy and want your (*table*) to

Is there really such a thing as a "self-reflective process of discovery"? (Google says no.)

- write two problems on any subject that has cropped up this year, and
- write a potential title for Day 14's problem set.

Hey, creativity counts! We'll include many of the best problems on Day 14's set, and the best title. Feel free to poke fun at easy targets like Art or Steve.

Your table's problems will be judged on a sliding scale that includes "Important", "Neat", "Wacky", and "Useless", among others.

Tough Stuff.

14. Break $x^5 - 1$ into quadratic factors, then use the factorization to *construct* a regular pentagon with straight edge and compass (or, use the Sketchpad tools).

15. Investigate the behavior of the function $z \mapsto z + \frac{1}{z}$ in the complex plane. An interesting paper on this and related topics can be found online at:

http://jwilson.coe.uga.edu/olive/Joukowski.Web/Joukowski.Paper.html

This paper includes some references to previous works by some guy from Washington.