No mathematicians were harmed during the making of this course.



# 14 *Escape From PCMI*

**Our Stuff.**

**0.** Is there any reason that a polynomial of degree $n$ can't have *more* than $n$ roots?

**0.** So, a cubic equation has at least one solution. Explain how factoring could help you show that there are a total of three solutions to the cubic equation.

Or, "The PCMI Redemption." Or, "Raiders of the Lost Arctan." Or, "2006: A PCMI Odyssey." Or, "$i$, Robot." Or, "A PCMI Too Far." Or...

Here a "repeated root" might be counted twice or more.

**Polynomial Stuff.**

**1.** Find all solutions to each quadratic equation.
(a) $w^2 - 26w + 169 = 0$
(b) $w^2 - 26w + 168 = 0$
(c) $w^2 - 26w + 167 = 0$
(d) $w^2 - 26w - 1 = 0$
(e) $w^2 - 26w + 170 = 0$

**8, 11.** Let $f(x) = -8x^4 - 9x^3 + 4x^2 + 3x + 8$ and $g(x) = 9x^3 - x^2 - 6x - 2$. Find the remainder when each of these is divided by $x^2 + 1$.
(a) $f(x)$
(b) $g(x)$
(c) $f(x) + g(x)$
(d) $f(x) - g(x)$
(e) $g(x) + 19$
(f) $f(x) \cdot g(x)$

**9.** (a) Construct a quadratic equation with roots $2 + i$ and $2 - i$. (Sum and product?)

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(b) Harder: construct a quadratic equation with roots $2+i$ and $-2+i$.

**7.** The graph of $y = x^2 + 10x + 27$ is a parabola. Look back at problem 14 from Day 8 and answer the same questions about this graph.

**Geometric Stuff.**

**11.** Use Sketchpad to build a triangle $OIL$. Then plot each of these:

- the *orthocenter*, intersection of the altitudes
- the *centroid*, intersection of the medians
- the *incenter*, intersection of the angle bisectors
- the *circumcenter*, intersection of the perpendicular bisectors

Which one is the odd man out?

Since all three of each type intersect in the same point, you only need to plot two, then find the intersection. Whoomp, there it is.

**8.** Triangle $A(2,3), R(4,7), T(-5,1)$ needs to transform into triangle $P(8,18), E(18,40), G(-3,-11)$.

(a) Write a transformation in the form $(x,y) \mapsto (ax+by, cx+dy)$ that does this.

(b) Is there only one such transformation?

**2.**
- Cut a strip of paper about 1.5 cm wide.
- Tie a knot near the end of the strip.
- Flatten everything to form a polygon. Where have you seen this shape? (Powers of $z$?)
- Fold the strip around the polygon.
- Tuck the end in.
- Puff up the shape by pressing the midpoint of each side using a fingernail.

Our hope is that Table 2 will prepare an example of this construction.

**1.** Slider Man, governor of Hyperbolia, wants to build thhe WTC (Wormhole Transit Center) somewhere between the three cities of Hyperion, Balloonium, and Averium. (Triangle BAH?)
Given the space is hyperbolic, where should the WTC be built so as to minimize the total length of the three new geodesic roads?

This problem seems familiar somehow, I just can't place it...

**6.** Does the Pythagorean Theorem carry over into the hyperbolic plane? What about the Side-Angle Inequality (longest side opposite longest angle)?

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**Complex Stuff.**

**8.** Describe in detail how the locus sketch is used to solve equations in the complex plane. Include:
(a) What's the meaning of the green circle?
(b) What's the meaning of the red locus?
(c) Which is first, magnitude or direction of solutions?
(d) Why wouldn't we do that in reverse order?

**5.** (a) Find the 12 solutions to $x^{12} - 1 = 0$.
(b) If one vertex of a regular dodecagon is $(1, 0)$ and the center of the dodecagon is $(0, 0)$, what are the others?

**9.** How many complex numbers $a + bi$ have integer $a, b$ and magnitude $\sqrt{325} = 5\sqrt{13}$? What of $\sqrt{1105}$?

**10.** Approximate the three solutions to the equation

$$x^3 + ix^2 - 2x = i$$

**3.** Solve $z^5 + 2z^4 - 3z^3 + 2 = 0$ in any way you can. (Must do it graphically.)

**4.** (a) Take a complex number $z$, then continue squaring it. What could happen, depending on what $z$ is?
(b) (Tougher.) Start with a complex number $c$. Square it, then add $c$. Then square that, and add $c$. Keep doing that. What could happen, depending on $c$?

**Wacky Stuff.**

**0.** What number is this?

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}}$$

**0.** What number is this?

$$2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \ddots}}}}}$$

Here the numbers on the outside go $2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, \ldots$.

**7.** Amanda's favorite function is $A(x) = x^4 - 3x^2 + x^2 + 2$. Find a cubic function, a quadratic function, and a linear function that are all tangent to $A(x)$ at $x = 1$.

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**10.** (a) Find a third or fourth-degree polynomial tangent to $y = x^5 - x^3 + 3x - 7$ at $x = -1$ *and* $x = 2$.

(b) Find the one fourth-degree polynomial that is tangent at $x = -1$ and $x = 2$, and also passes through the point $(0, -7)$.

**11.** Consider the following:

$$7^2 + 21^2, 9^2 + 27^2, 13^2 + 39^2, \ldots, n^2 + (3n)^2$$

$$7^2 + 14^2, 9^2 + 18^2, 13^2 + 26^2, \ldots, n^2 + (2n)^2$$

Do you notice any patterns? Can these patterns be used to generate Pythagorean triples?

**11.** Check these out:

$$45^2 + 46^2 = ?, 37^2 + 67^2 = ?, 53^2 + 25^2 = ?$$

Is there a pattern? Can you generalize or prove it?

**5.** How many Reidemeister moves does it take to untangle $f(z) = z^5 + z^4 + z^3 + z^2 + z + 1$ where $z$ is a complex number on the function $f(x) = \cos x$? Assume that the ends of $f(z)$ are connected by a straight line.

What's a Reidemeister move, you say? And how do we know which parts of the path are over or under other parts? Uhhh.

**8.** Extend the sketches from the third week to allow for magnitude circles that aren't centered at the origin. Try this with $f(z) = z^3 + 1$ and see what happens. (Where might be a good place to center the circle, then?)

**Tough Stuff.**

**12.** Build a polygon in the complex plane, then a point outside the polygon. Using locus constructions, divide the point into the polygon. What could the resulting shape look like for different polygons?

You will need several loci, one for each segment. Or try merging? Ben might be lying about this.

**5.** Consider the dodecagon from problem 14. If each vertex of the dodecagon is transformed by the rule

$$(x, y) \mapsto (3x - 2y, 4x + 5y)$$

find the exact area of the new shape formed.

**6.** Assuming that Matsuura triangles exist, prove that one side length of the triangle must be a multiple of 24.

Thanks, everyone! It's been a real treat teaching the class and meeting you all. See you again soon.