2007.12 The Crying Game

Game of the Day: "Who Wants To Be A Millionaire?"

In 2000 it was on four times a week, and it was routinely the #1 through #4 shows in the ratings. On the show, multiple-choice questions with four options are presented.

Question 1 was usually really easy, but it didn't stop someone from saying that Hannibal crossed the Alps on llamas. *Oops!*

Picture yourself on Millionaire and totally spooked by the cameras. You guess at each question. Ignore (for now) the fact that if you get a question wrong, you're off the show.

Well, you got the first question wrong . . . here, have another chance!

1. You get asked two multiple-choice questions and take a guess at each. Find the mean and variance for the *number of questions* you will get right. There are 16 possible outcomes here, and they are

$$0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 2$$

Are you sick of this particular fraction yet? It sure comes up often.

2. Same for three questions: what is the mean and variance for the number of questions you will get right?

Say, this would be a pretty good $1 million question.

3. Expand the polynomial $(r + 3w)^4$ and use it to find the mean and variance for the number of questions you get right in four tries. Or use your own method.

Some Millionaire-style math problems are available at www.ams.org/wwtbam/archive.

Important Stuff.

4. Pick two positive integers at random. What is the probability that they do not share a common factor greater than 1? Give your answer to six decimal places.

Here's a table of the 36 ordered pairs of numbers (x, y) with $1 \leq x, y \leq 6$.

Looks like what you could get from rolling two dice to me.

$(1,6)$	$(2,6)$	$(3,6)$	$(4,6)$	$(5,6)$	$(6,6)$
$(1,5)$	$(2,5)$	$(3,5)$	$(4,5)$	$(5,5)$	$(6,5)$
$(1,4)$	$(2,4)$	$(3,4)$	$(4,4)$	$(5,4)$	$(6,4)$
$(1,3)$	$(2,3)$	$(3,3)$	$(4,3)$	$(5,3)$	$(6,3)$
$(1,2)$	$(2,2)$	$(3,2)$	$(4,2)$	$(5,2)$	$(6,2)$
$(1,1)$	$(2,1)$	$(3,1)$	$(4,1)$	$(5,1)$	$(6,1)$

5. Look for a pattern to these answers.

(a) What fraction of these ordered pairs have *both* numbers divisible by 2?

How can there be a pattern when there's only one question?

(b) What fraction of these ordered pairs do *not* have both numbers divisible by 2?

Oh. Two pages.

(c) What fraction of these ordered pairs have *both* numbers divisible by 3?
(d) What fraction of these ordered pairs do *not* have both numbers divisible by 3?
(e) Cross out any of the 36 ordered pairs where both numbers are divisible by 2. What fraction of the original 36 ordered pairs remain?

This is the obligatory reference to Christopher Cross and/or the unrelated duo Kris Kross. This problem is totally krossed out!

(f) Now cross out any remaining ordered pair where both numbers are divisible by 3. What fraction of the numbers that survived part (e) also survived this second cut?
(g) What fraction of the original 36 ordered pairs survived both cuts?

6. Build a grid with the numbers 1 through 15 as labels on the sides. You now have 225 ordered pairs.

(a) How many of the 225 ordered pairs do *not* have 3 as a common factor (in both numbers)? For example, $(9, 6)$ has 3 as a common factor, but $(10, 12)$ does not.
(b) How many of the 225 ordered pairs do *not* have 5 as a common factor (in both numbers)?
(c) How many of the 225 ordered pairs do *not* have either 3 or 5 as a common factor (in both numbers)?
(d) If Gloria picks an ordered pair at random, what is the probability that it does *not* have either 3 or 5 as a common factor (in both numbers)?
(e) Multiply this out:

$$\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{5^2}\right)$$

7. Calculator skill time. Using a calculator page, it is possible to define custom functions.

- Get a new calculator page by hitting HOME, option 5 (New Document), then selecting Calculator.
- Hit MENU, then select option 1 "Tools", then option 1, "Define".
- Type: $f(n) = 1 - \frac{1}{n^2}$. Your calculator line should look like

You could also type out the word d-e-f-i-n-e, then a space (next to Z). Nah.

$$\text{Define } f(n) = 1 - \frac{1}{n^2}$$

- Hit enter, and watch no magic as the calculator just says "Done".
- Now type $f(3)$ and hit enter. Hey hey! I feel a sense of deja vu.

It's either deja vu or something I ate.

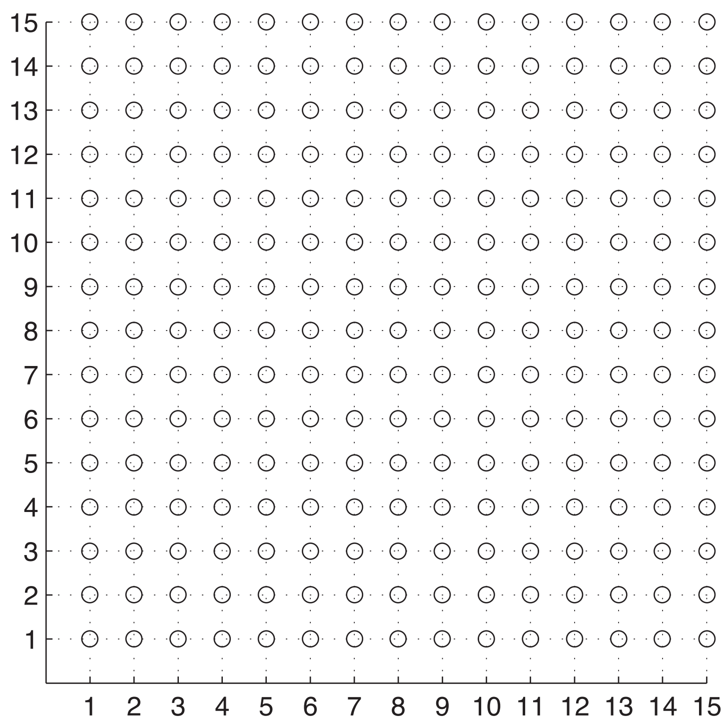
Find this product using your new function:

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{5^2}\right)$$

and give your answer as a decimal using the blue **ctrl** button.

Will you still need me, will you still feed me... when I'm...

8. Here is a coordinate plane, 15-by-15, with an open circle at each point (x, y) with integer coordinates.



(a) Using a red pen, color in the circles for any point where both x and y share a common factor of 2.

(b) Using the same pen, color in any additional circles for any point where both x and y share a common factor of 3.

(c) Using the same pen, color in any additional circles for any point where both x and y share a common factor of 4. Oh, okay. Well, that was easy. Next page!

What, you ate your red pen? Fine, do what you like! But use the same easily seen color.

(d) Using the same pen, color in any additional circles where both x and y share a common factor of 5.
(e) Do it again for 7; for 11; for 13. Oh my.

Neat Stuff.

9. Calculate this product as long as necessary:

$$\prod_{p \text{ prime}} \left(1 - \frac{1}{p^2}\right)$$

The giant π just means multiply, so this is the same exact thing as

$$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{5^2}\right)\left(1 - \frac{1}{7^2}\right)\left(1 - \frac{1}{11^2}\right)\cdots$$

where each denominator is a prime. Say, I think you have a calculator function to simplify this work. When you multiply all these "slightly less than 1" things together, what happens?

Mmmm, giant pie. Man, there's a lot of food references today. By the way, it's sigma for sum and pi for product, if that helps to remember them.

This is a pretty interesting result, though, wicked awesome as they say. Take enough terms until you're happy. The primes are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, and maybe a few more.

10. A parallelogram has vertices $(0,0)$, (c,a), (d,b), and $(c+d, a+b)$. Find its area in terms of the variables given. Also, find the area of a few parallelograms formed by consecutive "Farey fractions". Take your pick.

This is not a subtle reference to Pick's Theorem. But, that would've been a good idea.

11. Ah, back to Millionaire. Let's say that you're not totally terrible at answering the questions. You can answer any of the first 5 questions with 90% probability, any of the middle 5 questions with 75% probability, and any of the last 5 questions with 60% probability. Find the mean and variance for the number of questions you get right *before missing one*. Sorry, no lifelines!

Bookkeeping can be tricky here, so feel free to approximate a bit.

12. What's the probability that a positive integer is *square free*? That is, 4 is not a factor, and 9 is not a factor, and 16 is not a factor.... I guess we can skip 16 because of 4.

Tough Stuff.

13. Use Pick's Theorem to show that if $\frac{a}{c}$ and $\frac{b}{d}$ are consecutive "Farey fractions", then $bc - ad = 1$ every time.

Ask around about Pick's Theorem or look it up. It's a great Geoboard theorem.

14. Show that if $\frac{a}{c}$ and $\frac{b}{d}$ are consecutive "Farey fractions", then the fraction with least denominator between $\frac{a}{c}$ and $\frac{b}{d}$ is $\frac{a+b}{c+d}$.