2007.14 Game Over, Man!

Game of the Day: "Deuce"

John and Jim are playing tennis. They reach "deuce", a point in the game where the next two points can determine a winner:

- If John wins both points, he wins the game.
- If Jim wins both points, he wins the game.
- If the two points are split, the game returns to "deuce" and continues.

John is serving, so he wins each point with probability 0.6. Find the probability that John wins the game before Jim does.

There is no limit to the number of times the game reaches "deuce" again. The most deuces ever in a pro tennis game is 37.

Important Stuff.

13. Flip 4 coins and roll 2 dice. What is the probability of getting exactly 2 tails *and* a sum of 7 on the dice?

We have grouped the problems roughly by category. There is a purpose to the numbering. Flip around!

8. (a) Build a table for the 36 outcomes for rolling two standard dice.
(b) Find the mean and variance for the two dice rolls.
(c) Expand $(x + x^2 + x^3 + x^4 + x^5 + x^6)^2$. What do you notice?
(d) Find the mean, variance, and standard deviation for rolling four dice.

You found the variance for rolling one die on Day 13. Die, another day.

10. Given what you know now, how would you approach the question of determining if a series of 240 coin flips came from a fair coin or was faked?

8. You are working your way through the lunch line in the PCMI tent. It is sandwich day, and there are five stations where you may reach across the table or stay on your side for your preferred option (you pick one of each dressing, pasta, bread, condiment, and meat).
(a) What is the probability that you will get each of your preferences at all five stations if you stay on your side and *never* have to reach? You'd need some good luck . . .
(b) What is the probability that you have to reach to get your preference *every time*?
(c) Expand $(a + b)^5$. What's up with that?

The probability that Mitch gets his preferred dressing is zero, because they don't ever have bleu cheese!

Yes, this table actually wrote "What's up with that?" in their problem. Nice one.

(d) Use the expansion to find the probability that you have to reach *3 or more* times to get your preference.

9. In a semi-popular variant on an old game, Bear mauls Ninja, Ninja sneaks up on Cowboy, and Cowboy shoots Bear.

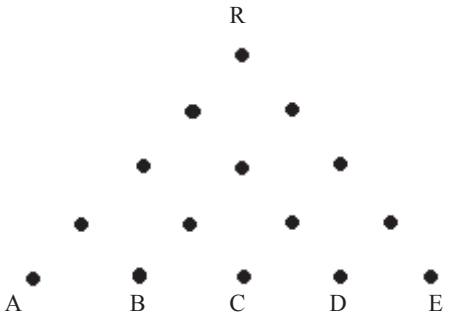
Avery just keeps picking ninja. He does it every time. Sendhil never wises up on this, though, and picks Bear and Cowboy with equal probability. Find the probability that

(a) Avery wins three games in a row
(b) Avery wins exactly two out of three consecutive games
(c) Avery wins a best-of-five tournament by winning 3, 4, or 5 games out of 5

Note that tables from opposite sides of the room came up with these last two problems. Best-of-five tournament? Five stations? Hm.

8. Ralph makes like a Plinko chip and percolates in the random walk during the Fourth of July Parade. If Ralph starts at point R and can move downward either to the left or right each time to the next available row, what is the probability that Ralph ends up at point C?

In this process, Ralph runs into several people, not just Pegs.



6. Start at the origin and flip a coin. If it comes up heads move 1 unit to the right and 1 unit up. If it comes up tails, move 1 unit to the right and one unit down. Repeat this for 3 more flips (4 flips total).

(a) How many different paths end at the point (4,2)?
(b) Do any paths pass through (2,1)? Why or why not?

10. Find the fraction with smallest possible denominator between $\frac{3}{5}$ and $\frac{2}{3}$.

2. To supplement funding, PCMI holds a cow-pie derby. A 50-by-50 grid is marked and squares are sold. Since tickets

Cow pies are not square, nor do they easily divide by 6.

are sold only to mathematicians, they buy only squares with coordinates that have no common factors greater than 1. What is the probability that PCMI will *not* need to pay a prize? (Assume all possible tickets are sold, and that only one cow-pie is dropped.)

10. Pick two integers with $x \leq y$ and $1 \leq x, y \leq 15$.
(a) Find the probability that the fraction $\frac{x}{y}$ is already written in lowest terms.
(b) Suppose we allow $x > y$ as well. What happens now?

Consider $\frac{9}{7}$ to be in lowest terms for the purposes of this problem, even though it is an improper (sometimes called "vulgar") fraction. Proper!

12. (a) For each element $\frac{y}{x}$ of F_6, the Farey sequence of order 6, plot the point (x, y).
(b) Find all points that intersect the line $y = x$.
(c) Find all points that intersect the line $y = x - 1$.
(d) Find all points that intersect the line $y = \frac{1}{2}x$.
(e) Find all points that intersect the line $y = 2x$.

11. (a) Imagine a golden spatula. Now ignore that and draw the graph of all numbers with $1 \leq x, y \leq 6$ with $x > y$ such that x and y have no common factors.
(b) Do you have all the elements of F_6? Explain. Why does this work?
(c) Think about it: what has your spatula learned?

If this doesn't make sense, don't blame us!

13. How many *more* fractions does the Farey sequence F_{210} have than the Farey sequence F_{209}?

A lovely problem from clap clap clap clap deep in the heart of Texas.

12. For each Farey sequence F_3 through F_6, find the mean, variance, and standard deviation of the numbers in the sequence. (For F_3 this is $0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1$.) Any patterns or thoughts?

"You stink" does not qualify as a reasonable "thought" on this or any problem.

4. Angie is playing with a single suit of cards. Each card gets a value. Kings, queens, jacks, and tens are worth ten each. All other number cards (2 through 9) are worth 5. The ace is worth -15. So, don't draw the ace.
(a) Find the mean, variance, and standard deviation for 1 pick.
(b) Find the mean, variance, and standard deviation for 2 picks, drawing with replacement.
(c) Find the mean, variance, and standard deviation for 65 picks, drawing with replacement.
(d) Write an expression you could expand to help with this situation.

Shouldn't 64 picks work out more nicely? Apparently not!

11. Three people are picked for the Big Lotto Spin. The wheel contains 3, 5, 7, 10, and 15 million. Each contestant get to spin and win three times!

This puts "Who Wants To Be A Millionaire" to shame. Minimum prize 9 million! Where do I sign up?

(a) Find the mean, variance, and standard deviation for one spin.
(b) Find the mean, variance, and standard deviation for two spins; for all three spins.
(c) How much money, on average, will the show give out in one episode (with three players spinning three times each)?

12. Using the nSpire, find the mean, variance, and standard deviation for this data set:

$$2, 7, 8, 9, 10, 15, 20, 22$$

11. The marginally popular spin-off show Wheel of Lures is played with a five-stop wheel. Its values are $-1, 3, 5, 7, 11$.

We changed the 8 to a 7. Hope you don't mind. Oh, and we changed the 10 to a 9 in the second set. Positive integers are our friends, naturally.

(a) Find the mean, variance, and standard deviation for one spin of the Wheel.
(b) The game also uses an eight-sided die with the sides $-3, 4, 5, 6, 7, 9, 12, 16$. Find the same stats for the die.
(c) In the Final Countdown round, the wheel is spun twice and the die is rolled once. Find the variance for this round.

Sing along! *It's the final countdown!*

(d) Write an expression that, if expanded, would list all 200 outcomes for the Final Countdown round.

1. On the new game show "Prime Time," the lucky contestant spins a wheel with seven equal spaces, numbered 2, 3, 5, 7, 11, 13, and 17, which determines their winnings.

Prime Time is sponsored by Seventeen magazine, hence the top value 17. The note from table 1 says: "Bowen has never appeared in Seventeen Magazine. He was, however, in Tiger Beat after his pinball championship last year." Bowen replies: not true. It was Bikini Magazine.

(a) If you randomly select two numbers from the wheel, what's the probability they have no common factor greater than 1?
(b) Find the mean, variance, and standard deviation of their winnings on one spin, two spins, three spins.

1. After their sponsorship contract ends, the producers decide that there's not enough conflict in the show and start recruiting failed contestants from reality shows and rename the show "What's My Number Line?" They decide to change the numbers on the wheel as well, to $-7, -6, -4, -2, 2, 4$, and 8.

(a) What do you notice about these numbers compared to the old ones (2, 3, 5, 7, 11, 13, 17)?
(b) Find the mean, variance, and standard deviation of their winnings on one spin of this new wheel.
(c) How do your answers compare to the previous problem? Any conjectures?

6. In Mr. Bolognese's class, the students can choose 4 numbers out of 1-10 to make a Homework Lottery Ticket.

Gambling is illegal at Bushwood. I hope the parents don't find out about this program!

(a) How many possible Homework Lottery Tickets are there? Order doesn't matter. Use Pascal's Triangle to help.
(b) Chris then picks four numbers from 1 through 10 and awards homework passes for each matching number. Find the mean number of passes a student can expect to earn from this game.

9. Phil and Deb are lost in a game of D&D (Dungeons and Dragons). The game uses five dice, one for each Platonic solid: 4 sides, 6 sides, 8 sides, 12 sides, 20 sides. Each die is labeled from 1 to its total number of faces (like normal, so the cube is 1-6). Phil rolls all five dice.

I don't know why Phil didn't just roll 2d20 and cast a Spell of Remembrance.

(a) For each die, calculate the mean and variance for the number rolled. You might want to break this up among a group!
(b) Find the mean, variance, and standard deviation for the sum of the results on all five dice.

7. Use all of the following concepts in a cohesive mathematical narrative.

* shrug * I guess read it and move on? It's funny at least.

- $\sum \frac{1}{n^2}$
- integers with no square factors
- $\frac{3}{\pi^2}$
- points in the first quadrant visble from the origin
- $\frac{\pi^2}{6}$
- fractions in the Farey sequence visble from the origin
- $\frac{6}{\pi^2}$

Neat Stuff.

10. (a) Wheel A has the following outcomes: $1, 3, 5, 7, 9, 11, 13$. What is the mean, variance and standard deviation for the outcomes of one spin?

(b) Wheel B has the following outcomes: $0, 2, 2, 8$. What is the mean, variance and standard deviation for the outcomes of one spin?
(c) The object of a game is to spin both wheels and subtract the value of wheel B from wheel A. The contestant either earns or pays (if the result is negative) the amount. What is the mean, variance and standard deviation for the outcomes of one round?
(d) What might the product of two polynomials look like that could be used to find the outcomes of this game?
(e) Suppose 5 rounds are played. What is the expected amount of money earned in the 5 rounds? Use the result from 1 round to help.

5. There are two red fish and three blue fish in a fish tank. Tired of thinking about fish, you randomly pull out a fish one at a time and flush it down the toilet. You only stop after "eliminating" both of the red fish.

So, is this a Poisson distribution, or a distribution of poisson?

(a) What is the probability that you will stop after flushing the third fish?
(b) How many fish do you expect to flush overall?
(c) Generalize to r red fish and b blue fish.

5. Nicole and Darryl are the finalists in the International National Cowboy/Ninja/Bear Championship. A player wins the championship after winning 3 showdowns. What is the expected number of showdowns before a winner is determined? Remember, ties can occur.

For the rules, see earlier in the problem set. Like the Olympics, this event is held only every four years. Or, it seems, every night around these parts.

5. Jim and Bree play the following coin game. Jim is given the sequence of heads and tails THH. Bree is given the sequence HHH. The winner is the player whose sequence appears first. For example, if the sequence of coin flips were HHTHTTHH, Jim would win.

Jim must've set up these rules . . .

(a) What is the probability of Bree winning this game?
(b) How would the game change if Jim started with THH and Bree started with HTH?

13. In the sequence of lights with a 30-by-30 grid, what would be the first 5 lights *blocked* by other lights, panning from east to north?

2. PCMI participants receive grades for their performance on the problem sets. The grade comes from 3 options:

- Roll a die with six grades: A, B, C, C, D, F.
- Start with 75, then toss a coin. Heads gives 80 for a B, tails gives 70 for a C.
- Start with 50, and spin the Wheel of Fish five times. The number of fish is added to your grade.

Which of these gives the best strategy? (Does it depend on what you're after?)

Apparently, by asking for ¡No mas pescados! we were really saying "Only fish! Give us just fish! We love fish! Yes!" So says Table 9.

4. Bag 1 contains 3 green balls and 2 yellow balls. Bag 2 contains 2 green balls and 5 yellow balls. A ball is drawn at random from Bag 1, then placed into Bag 2. What is the probability of now drawing a green ball from Bag 2?

And after all this, who's left holding the bag?

Tough Stuff.

8. (a) Give a precise definition for the totient function $\phi(n)$.
(b) Calculate $\phi(4)$, $\phi(9)$, and $\phi(36)$.
(c) If a and b are positive integers, under what conditions does $\phi(a) \cdot \phi(b) = \phi(ab)$?

ϕ'd me, Seymour!

7. From the origin, look down the x-axis and slowly turn counterclockwise. We explored earlier where you are facing when you have see half of the total visible points. What fraction have you seen when your gaze is described by $y = nx$ for a natural numbers n? Use some of the colored grids you have made, then try to generalize.

7. For curious calculus afficionados. There is a connection between the derivative of a generating polynomial and the mean. From yesterday's "Press Your Luck" game,

$$\frac{d}{dx}\frac{(x^3 + x^6 + x^7 + x^9 + x^{10} + x^{-5})^2}{6^2}$$

gives ten when setting $x = 1$. Ten is the mean of rolling two Press Your Luck dice.

(a) Explain why this works.
(b) Find a connection variance and the second derivative.

0. Explain why $e^{\pi\sqrt{163}}$ is an integer. Amazing!

0. Suppose you were allowed to continue rolling the "Press Your Luck" die with sides 3, 6, 7, 9, 10, and one Whammy side, earning points until you either decided to quit or until you hit a Whammy. This time, the Whammy behaves like it should and takes *all* your earnings. So, when should you be willing to quit the game?