2007.1   Head Games

Hey, welcome to the class. We know you'll learn a lot of mathematics here—maybe some new tricks, maybe some new perspectives on things with which you're already familiar. A few things you should know about how the class is organized:

- **Don't worry about answering all the questions.** If you're answering every question, we haven't written the problem sets correctly.

- **Don't worry about getting to a certain problem number.** Some participants have been known to spend the entire session working on one problem (and perhaps a few of its extensions or consequences).

One question from a previous year turned out to be the unsolved Twin Primes Conjecture. Nobody got that one.

- **Stop and smell the roses.** Getting the correct answer to a question is not a be-all and end-all in this course. How does the question relate to others you've encountered? How did others at your table think about this question?

- **Teach only if you have to.** You may feel the temptation to teach others at your table. Fight it! We don't mean you should ignore your tablemates but give everyone the chance to discover. If you think it's a good time to teach your tablemates about integration by parts, think again: the problems should lead to the appropriate mathematics rather than requiring it. The same goes for technology used in the course. The problems should lead to the appropriate use of technology rather than requiring it.

- **Each day has its Stuff.** There are problem categories: Game of the Day, Important Stuff, Neat Stuff, and Tough Stuff. We'll have you do Game of the Day first thing (most of the time), but after that you should check out Important Stuff first. We try to make sure that the problems in Important Stuff can be picked up regardless of how much or little work you've done on prior sets. All the mathematics that is central to the course can be found and developed in the Important Stuff. Hey, it's important stuff. Everything else is just neat or tough.

On Day 3, go back and read these again.

Will you remember? Maybe!

And now, on with the show . . .

Game of the Day: "Fake It, Make It"

1. With a partner or two, *fake* the results of flipping a coin 240 times. Write heads as 1, tails as 0. Seriously, fake it: no technology, no dice, nothing.

2. Now, with the same partnerortwo, *make* the results of flipping a coin 240 times. Write heads as 1, tails as 0. Seriously, make it: no technology, no dice, just flip a coin 240 times and write down what it says. Identify the two lists in some way that you would recognize, but a neighbor would not.

3. Create a test you could use to decide whether a list someone gave you was real or fake. Now exchange lists with a neighbor and run your test. Did it work?

Important Stuff.

4. (a) Pick two integers between 1 and 5 (inclusive). Find the probability that the two numbers picked do not share a common factor greater than 1.
   (b) Repeat for picking between 1 and 6, 1 and 7, 1 and 8, 1 and 9.

Be clear about any assumptions made in how you chose to do the picking—there is more than one way to do it, and they will yield different results that we'll talk about.

5. (a) If you flip a fair coin 240 times, how many heads would you expect?
   (b) Take a guess: what is the probability of getting *exactly* this many heads?

Don't take more than 10 seconds to guess!

6. (a) If you roll a fair die 240 times, how many ones would you expect?
   (b) Another guess: is it more likely to get exactly this many ones, or to get exactly the number of heads from the last problem?

Dude, just take a guess and move on.

7. Jocelyn has a piece of paper. She tears it into three equal pieces and hands one piece to Connie, another to Mario, and keeps the third piece for herself.

   She continues to do this; tearing the paper she has left into three equal pieces, handing one piece to Connie, one to Mario, and keeping the third.

   (a) After two tears, how much paper does Jocelyn have left? How much do Connie and Mario each have?
   (b) After three tears?

    (c) After four tears?

    (d) After 10 tears?

    (e) Forever?

    (f) Write two different expressions for the amount of paper Connie has after this is over.

8. Mary has a piece of paper. She tears it into four equal pieces and hands one piece to Manuel, one piece to Kim, one piece to Sandra, and keeps the fourth piece for herself.

   She continues to... aw, do we have to write the rest of this? Fine.

       (a) After two tears, how much paper does Mary have left? How much do Manuel, Kim, and Sandra each have?

       (b) After three tears?

       (c) After four tears?

       (d) After 10 tears?

       (e) Forever?

       (f) Write two different expressions for the amount of paper Sandra has after this is over.

Neat Stuff.

9. What's the probability that an integer picked from 1 to n is a perfect square if

       (a) $n = 10$?

       (b) $n = 100$?

       (c) $n = 1000$?

       (d) $n = 10000$?

       (e) What is happening "in the long run" (as n grows larger without bound)?

10. A gambler offers you these two games:

    Game 1: You roll a die four times. If you hit a 1 any of the four times, you win.

    Game 2: You roll a pair of dice 24 times. If you hit double 1s any of the 24 times, you win.

    Aside from the length-of-game issue, which of these games would you rather play? Or are they the same?

11. The new toy craze is Mega Men, where kids buy a Mega Man in a box without looking to see which one it is, then open it up when they get home. There are ten toys in all, each equally likely when you buy a box. If you collect all ten, you can make Super Ultra Mega Man!

On average, how many boxes will you have to buy for your kid before he can finally collect them all?

12. Okay, so there's this game. You get 1 point every time you flip heads. But, anytime you flip tails you're in "danger". If you flip tails a second time *consecutively*, you "bust" and lose all your points (but continue playing).

    (a) The game lasts 10 flips. What is the probability that you survive all 10 flips without busting even once?

    (b) What is the average score you could expect after 10 flips?

    (c) What happens in a longer game? Will the average score increase? Is there a limit?

Sadly, eBay is not an option, since the only cool Mega Men are the ones still in their original packaging.

So, don't flip tails twice in a row. Otherwise it's all good.

Tough Stuff.

13. Remember Yahtzee? Sure, you remember Yahtzee. You get three rolls and you're looking to get all 5 dice to be the same number. You can "save" dice from one roll to the next. There are other goals, but people really only care about getting the 5-dice Yahtzee.

    (a) Find the probability that if you try for it, you will get a Yahtzee of all 6s in any given turn.

    (b) Tougher: find the probability of getting any Yahtzee by trying for it—that is, you always play toward the nearest available Yahtzee.

14. Build a data set with at least 5 elements such that if m is the mean and n is the median, then $|m - n|$ is larger than the standard deviation of the set.

Since this is Tough Stuff, we don't have to tell you how to calculate standard deviation.