"... in the end i know, but on the way i wonder." — Cat Stevens

7 *i of the T(r)i(an)g(ul)er*

Important Stuff

PROBLEM

Here are some number facts that we've seen over the last few days.

$$N(2+i) = (2+i)(2-i) = 5 \quad \text{and} \quad N(3+i) = (3+i)(3-i) = 10$$
$$N(2-i) = (2-i)(2+i) = 5 \quad \text{and} \quad N(3-i) = (3-i)(3+i) = 10$$

Use these facts to come up with some complex numbers $x + yi$, with x and y integers, that have $N(x + yi) = 50$? How many complex numbers $x + yi$, with x and y integers, have $N(x + yi) = 50$? When you plot all of these numbers on a complex plane, what connections can you make? What happens when you take into consideration that $N(1 + i) = 2$ and $N(1 - i) = 2$? What if we want numbers that have $N(x + yi) = 250$ instead?

"Just the facts," said Joe on Friday.

1. If you didn't get to this problem yesterday, please try it today. If you did, please look at it again.
(a) Show that $z^3 - 1 = (z - 1)(z^2 + z + 1)$.
(b) Find all three solutions to the equation $z^3 = 1$.
(c) Plot all of your solutions carefully on the complex plane. You may need to use a calculator to help you find their locations on the plane.

2. Let $w = \dfrac{-1 + i\sqrt{3}}{2}$. Use exact arithmetic to calculate these, then plot them. You may need to convert some numbers to decimals to plot them accurately.

Or maybe you just need some other kind of graph paper.

(j) w^2
(a) w^3
(n) w^4
(i) w^{299}
(e) w^{300}
(c) w^{301}
(e) $(w^2)^3$ (So, what equation does w^2 solve?)

Darryl wants a Nintendo wi^{102}!!!

3. Let $a = 4 + 0i$, $r = 2 + 0i$, $t = 4 + 4i$, and $w = \dfrac{-1 + i\sqrt{3}}{2}$.
(a) Plot a, r, and t in a complex plane and connect the points to form a triangle.
(b) Multiply t, a, and r by $-i$ and plot those points in the same plane.
(c) Multiply r, a, and t by w and plot those points in the same plane. You may need to convert some numbers to decimals to plot these points accurately.
(d) What's going on here?

4. On Day 3, we mentioned that $3+2i$ is the *conjugate* of $3-2i$ and vice versa. From now on, let's use a bar over number as the symbol for the conjugate of a complex number. For example, $\overline{3 + 2i} = 3 - 2i$.
(a) If $z = a + bi$, what is $\overline{z}$?
(b) What is $\overline{\overline{z}}$, the conjugate of the conjugate?
(c) What's $\overline{2}$?
(d) Explain why $N(z) = z\overline{z}$.
(e) How does the location of a number in the complex plane compare with the location of its conjugate?

Sometimes, this is called the *complex conjugate*.
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Hey there! Did you know $2 = 2 + 0i$?

5. Let $g = 5 + 2i$ and $h = 3 - 4i$. Calculate these.
(a) $\overline{g + h}$
(b) $\overline{g} + \overline{h}$
(c) $\overline{gh}$
(d) $\overline{g}\,\overline{h}$

6. Let $w = \dfrac{-1 + i\sqrt{3}}{2}$. Calculate these.
(a) $\overline{w}$
(b) $w + \overline{w}$
(c) $w\overline{w}$
(d) $(7 + 4w)(7 + 4\overline{w})$

Just let it. Come on, you know you want to.

7. Put the following points on the isometric dot plane in order of their distance from the origin, from closest to farthest.
(a) $J = [\rightarrow 7, \nwarrow 4]$
(b) $O = [\rightarrow 13, \nwarrow 4]$
(c) $C = [\rightarrow 4, \nwarrow 13]$
(d) $E = [\rightarrow 8, \nwarrow 5]$
(e) $L = [\rightarrow 0, \nwarrow 8]$
(f) $Y = [\rightarrow -4, \nwarrow 4]$
(g) $N = [\rightarrow -4, \nwarrow -4]$

$[\rightarrow 7, \nwarrow 4]$ means go 7 units to the right from the origin, then go 4 units in the "up-and-left" direction.

8. Plot each of the following points in a complex plane. You may need to convert some numbers to decimals to plot them accurately.

(m) $7 + 4w$
(e) $13 + 4w$
(l) $4 + 13w$
(a) $8 + 5w$
(n) $0 + 8w$
(i) $-4 + 4w$
(e) $-4 - 4w$

Yes, keep using the same value for w as you did on problem 6.

Neat Stuff

9. If $a^2 + b^2 = c^2$, why is the point $\left(\frac{a}{c}, \frac{b}{c}\right)$ on the unit circle?

10. Find the intersection(s) of the unit circle $x^2 + y^2 = 1$ and these lines.

(a) $y = 2x - 1$
(b) $3x - 2y = 2$
(c) $y + 1 = 4x$

11. Plot these points in a complex plane and figure out how far they are away from the origin.

(a) $3 + 4i$
(n) $\frac{3}{5} + \frac{4}{5}i$
(n) w
(a) $-5i$

What's with this w and why does it keep reappearing? Are you getting bored of it yet?

12. Let $p = a + bi$ and $q = c + di$, with a, b, c and d being arbitrary real numbers. Prove or disprove these statements.

(a) $\overline{p+q} = \overline{p} + \overline{q}$
(b) $\overline{pq} = \overline{p}\,\overline{q}$
(c) $N(\overline{z}) = N(z)$
(d) $N(p) = N(q)$ if and only if $p = q$
(e) If $N(p)$ is a multiple of $N(q)$ then p is a multiple of q
(f) $N(pq) = N(p)N(q)$
(g) $N(p + q) = N(p) + N(q)$
(h) $N(p^2) = N(p)^2$

13. What's $N(7 + 4w)$? Justify why your answer is correct.

14. Show that if $\left(\frac{a}{c}, \frac{b}{c}\right)$ is a point on the graph of $x^2+y^2-xy=1$ then $a^2 + b^2 - ab = c^2$.

15. What does the graph of $x^2+y^2-xy = 1$ look like anyway?

16. Remember how we defined N to be a function that takes a number and multiplies it by its conjugate? In $\sqrt{2}$-land, the conjugate of $a+b\sqrt{2}$ is $a-b\sqrt{2}$, so $N(a+b\sqrt{2}) = (a+b\sqrt{2})(a-b\sqrt{2}) = a^2-2b^2$. Calculate these.

(a) $N(5+2\sqrt{2})$
(b) $N(5-2\sqrt{2})$
(c) $(5+2\sqrt{2})^2$
(d) N(your answer to part c)
(e) $N(4+3\sqrt{2})$
(f) $(5+2\sqrt{2})(4+3\sqrt{2})$
(g) N(your answer to part f)

17. How many different peg-to-peg squares are there on an $n \times n$ piece of square dot paper?

Oh noes: starting with 3×3, there are bonus squares!

18. How many different peg-to-peg squares are there on an $n \times n$ piece of isometric dot paper?

19. How many different peg-to-peg equilateral triangles are there on an $n \times n$ piece of isometric dot paper?

Bonus triangles!!!

20. Find all points on the isometric dot plane, $[\rightarrow x, \nwarrow y]$, with x and y integers, that are $\sqrt{133}$ units away from the origin. What about $\sqrt{217}$?

21. Let $w = \dfrac{-1+i\sqrt{3}}{2}$. Use exact arithmetic to calculate these.

(a) w^{-1}
(b) $(\overline{w})^{-1}$

Tough Stuff

22. Using only geometry and no algebra (no quadratic formula, no completing the square), find all complex numbers a and b that solve

$$\begin{aligned} a+b &= -1 \\ ab &= 1. \end{aligned}$$

Absolutely trigoNOmetry!

23. Find a method that can produce infinitely many triples (a, b, c) with no common factors such that a triangle with sides a, b, and c can be drawn on isometric dot paper.

Drawn on isometric dot paper = two sides must lie along rows of dots and all three corners must land on dots. You might want to find one or two such triangles first!

24. Suppose n is a positive integer. Is there a right triangle with rational numbers as its side lengths with area n?