In this version of the movie, the main character will be played by a fish.

10 *Wall•i*

Important Stuff

1. (*i*) Use $4+i$ to generate a Pythagorean triple.
(*ii*) Use $4+w$ to generate an Eisenstein triple.
(*iii*) Get out yet another sheet of square/isometric dot paper. On the square dot paper, pick a point as your origin, and plot $4+i$ and $(4+i)^2$. Draw triangles that correspond to both numbers and label their side lengths. Turn the paper over and repeat for $4+w$ and $(4+w)^2$.

OK, these triples are supposed to be called *Eisenstein triples*. Bowen claims the "se" is silent. Eisenstein triples are positive integers with the property that they can be the side lengths of a triangle with a 60° angle. Eisenstein triples relate to triangles with a 60° angle in the same way that Pythagorean triples relate to triangles with a 90° angle. "Franz" Ferdinand Gotthold Max Eisenstein (1823-1852) was a student of Carl Friederich Gauss. We are also trying to go for the record for the longest side note here. Woo hoo!

PROBLEM

Something interesting happens when we raise $1+\sqrt{2}$ to different powers.

$$\begin{aligned}(1+\sqrt{2})^1 &= 1+\sqrt{2} && \text{and} && 1/1 = 1\\ (1+\sqrt{2})^2 &= 3+2\sqrt{2} && \text{and} && 3/2 = 1.5\\ (1+\sqrt{2})^3 &= 7+5\sqrt{2} && \text{and} && 7/5 = 1.4\\ (1+\sqrt{2})^4 &= \,?\end{aligned}$$

If you continue this pattern, what do you notice about the fractions? What's going on here? N-values anyone?

N is your friend. It takes a number and multiplies it by its conjugate.

In $\sqrt{2}$-land, the conjugate of $a+b\sqrt{2}$ is $a-b\sqrt{2}$, so $N(5+2\sqrt{2}) = (5+2\sqrt{2})(5-2\sqrt{2}) = 17$.

2. Here are some fun facts: $N(3+\sqrt{2}) = 7$ and $N(3+2\sqrt{2}) = 1$.
(a) Explain why $N((3+\sqrt{2})^2) = 49$.
(b) What's $N((3+\sqrt{2})^3)$?
(c) What's $N((3+2\sqrt{2})^n)$ for any positive integer n?

Fun is a relative thing.

3. Repeat the problem in the box, but start with these numbers instead of $1+\sqrt{2}$. What happens to the fractions?
(a) $3+2\sqrt{2}$
(b) $2+\sqrt{3}$
(c) $2+\sqrt{5}$
(d) $3+\sqrt{10}$

Get back to work!

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4. Find integers x and y with $x > 1$ so that $N(x + y\sqrt{7}) = 1$. Use these numbers to make a fraction that gives a really good approximation to $\sqrt{7}$.

Come on! You can make a better approximation than that! Ask Peg how to use the magic box.

5. If you are given a triple of numbers, how can you test whether it is a Pythagorean triple? An Eisenstein triple?

6. (a) What does the value of $N(5 + 2i)$ have to do with the distance from the origin to $5 + 2i$ on the complex plane?

(b) What does the value of $N(5+2w)$ have to do with the distance from the origin to $5 + 2w$ on the complex plane?

7. Why do we *square* numbers of the form $a + bi$ or $a + bw$ to obtain Pythagorean triples or Eisenstein triples?

Why *squaring* as opposed to taking a cube root, or reciprocal, or ⟨*insert obscure math operation here*⟩?

Old Stuff

8. Which positive integers n have we determined cannot be written as $x^2 + y^2$ with x and y being integers?

9. Which positive integers n have we determined cannot be written as $x^2 + y^2 - xy$ with x and y being integers?

Look at the problem in the box for Day 4. There may be other numbers that can't be written as $x^2 + y^2$ or $x^2 + y^2 - xy$ that we haven't identified yet.

10. What is i^4? What is w^3?

11. Find integers a and b so that $N(a+bw) = 7$. Find integers c and d so that $N(c+dw) = 13$. Calculate $(a+bw)(c+dw)$, then calculate $N((a + bw)(c + dw))$.

12. Find an Eisenstein triple (a, b, c) with $c = 91$.

Wait a minute! This number didn't go up!

13. Calculate the N-value of these numbers.

(a) $3 + 2\sqrt{2}$

(b) $3 + 2i$

(c) $3 + 2w$

14. Rewrite these without using fractions.

OK, OK. We know w has a fraction in it. That one doesn't count in your answer.

(a) $\dfrac{11 + 7\sqrt{2}}{3 + 2\sqrt{2}}$

(b) $\dfrac{17 + 7i}{3 + 2i}$

(c) $\dfrac{17 + 9w}{3 + 2w}$

(d) $\dfrac{290 + 190w}{29 + 19w}$

Neat But Potentially Useless Stuff

15. For each of the following values of n, find integers x and y with $x > 1$ so that $N(x + y\sqrt{n}) = 1$. Use these numbers to make a fraction that gives a really good approximation to $\sqrt{n}$.
(a) 6
(b) 11
(c) 17
(d) 529
(e) 85

Here are a bunch of Eisenstein triples. In other words, these numbers can be the lengths of a triangle with a 60° angle. The third number is the length opposite the 60° angle.
$(5, 8, 7)$ $(7, 15, 13)$
$(11, 35, 31)$ $(16, 21, 19)$
$(33, 40, 37)$ $(32, 77, 67)$
$(117, 40, 103)$ $(13, 48, 43)$
$(57, 112, 97)$ $(39, 55, 49)$
$(79, 79, 79)$
$(799, 799, 799)$
$(7999, 7999, 7999)$
Those last three are boring.

16. In an Eisenstein triple, why is the length of the side opposite the 60° angle usually in between the other two lengths? What kind of triangles are exceptions to this pattern? There are a bunch of Eisenstein triples over there. →

17. On Day 5, we noticed this algebraic identity having to do with Pythagorean triples and right triangles.

$$\left(m^2 - n^2\right)^2 + \left(2mn\right)^2 = \left(m^2 + n^2\right)^2$$

Is there an analogous identity for Eisenstein triples?

Refer to the whiteboard notes posted on our 2008 SSTP site. Convince yourself this identity is true (algebraically *and* geometrically) if you haven't already done so.

18. Pick some integers a and b and square $a + bi$ and $a + bw$ to generate some Pythagorean or Eisenstein triples. You may find that sometimes the numbers in the triple have a common factor. How can you choose a and b so that you get a primitive Pythagorean triple? What about a primitive Eisenstein triple?

In this context, a *primitive* triple of numbers is one that has no common factors.

19. Does squaring $a+bw$ give you all possible Eisenstein triples?

20. Figure out a way to generate triangles with integer side lengths that have a 120° angle.

21. Use what you know about the N function to factor $74+7i$.

22. Does anything unusual or special happen when you cube numbers of the form $a + bi$ and $a + bw$?

23. Plot the graph of $x^2+y^2-xy = 49$ on isometric dot paper.

24. Andrew says that he has an identity:

$$(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (bc + ad)^2.$$

Is this identity true? How is this identity related to complex numbers?

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Tough Stuff

25. Find a distance so that more than 24 points on a isometric dot plane are that distance away from the origin.

26. Find a distance so that more than 24 points on a square dot plane are that distance away from the origin.

27. Find a distance so that more than 24 points on *both* square and isometric dot planes are that distance away from the origin.

28. Derive the w-world version of the identity in problem 24.

29. Find two positive integers a and b such that when you make a right triangle with these two numbers as legs the hypotenuse is an integer, *and* when you make a triangle with a $60°$ angle with these two numbers as side lengths the side opposite the $60°$ angle is an integer. Or, prove that no such a and b exist.

Just understanding this problem statement is already an accomplishment... If you can rewrite this problem to make it easier to understand, please write it on the back of a $100 and give it to us.

30. The current record for calculating π is into the trillions of digits. One completely wacky formula for computing π comes from this calculation:

$$\frac{(Q+i)^{12} \cdot (57+i)^{32} \cdot (110443+i)^{12}}{(239+i)^5}$$

Oh wait, we forgot Q. Find the integer Q so that this entire number makes an $45°$ with the positive real axis when plotted in the complex plane. This formula was discovered by a high school teacher in 1982. Perhaps you could find some more?

Check out problem 19 on Day 5.