12 *Fantasy w-land*

Important Stuff

PROBLEM

Today's problem can't fit in this box.

Even if we make the box this big, it still doesn't fit. So, we're calling it the "problem out of the box." We promise it's still fun!

The charts with most the beautiful handwriting will get a surprise.

Since Part 2 of today's PootB looks just like yesterday's, you might want to write something on both pages to help you distinguish them.

1. What peg-to-peg distances are possible on a 6-by-6 piece of isometric dot paper?

Don't take more than three minutes on this problem.

2. On Part 1 of today's "problem out of the box," color in the circles according to the size of the number. For example, use a different color if the number is between (and including) 1 and 25, between 26 and 50, between 51 and 75, between 76 and 100. Count the number of circles in each category.

Stay within the lines!!

3. One way to identify (real) prime numbers is with the Sieve of Sergio. Here's how to do it.

- Cross out 1 because it's not prime.
- Circle 2, then cross out every multiple of 2.
- Circle the smallest number that hasn't been circled or crossed out yet (after circling 2 you would circle 3), then cross out all of its multiples. Some multiples may have already been crossed out—this is okay.
- Repeat the previous step until every number is either circled or crossed out. The numbers that are circled are prime.

OK, we know it's really the Sieve of Eratosthenes, but Sieve of Sergio sounds way cooler.

At what point can you stop this procedure and declare all remaining numbers prime?

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Neat Stuff

4. Look at the chart from Part 2 of today's "problem out of the box" to see what patterns you can find.

5. On a blank piece of square dot paper or grid paper, plot all the Gaussian integer multiples of $3+2i$ that fit on your page. Connect each multiple with its nearest neighbors. What do you notice?

This is similar to problem 2 from Day 11. $(3+2i)(1-i)$ is a multiple of $3+2i$.

6. Which of these numbers are prime in i-land? That is, which of these numbers cannot factor into Gaussian integers with N-value greater than 1?

(h) 11
(a) 13
(l) 19
(e) 37
(y) 41

7. Prove that if $z = ab$, then $N(z) = N(a) \cdot N(b)$.

8. Use problem 7 to prove that if the N-value of a Gaussian integer z is prime, then z is prime in i-land.

9. Which Gaussian integers are prime?

10. Which Eisenstein integers have N-value equal to 1?

Eisenstein integers, a.k.a. $w$integers, are numbers like $a + bw$ where a and b are integers.

11. Express $\overline{8 + 5w}$ as an Eisenstein integer. Write your answer like $a + bw$, so it doesn't have $\overline{w}$.

12. What's special about a triangle with sides 3, 5, and 7?

13. Which of these numbers are prime in w-land? That is, which of these numbers cannot factor into Eisenstein integers with N-value greater than 1?

(b) 3
(r) 5
(i) 13
(a) 29
(n) 43

14. Which Eisenstein integers are prime?

15. Pick an integer n with $2 \le n \le 12$. On Part 1 of today's "problem out of the box," use color pencils to color in the numbers according to their remainder after dividing by n.

Aren't you sad that there are no side notes on the last page of today's problems? Write some in.

16. For each of the integers below, make a list of all of its factors that are congruent to 1 mod 4 and 3 mod 4. Look for patterns that relate to Part 2 of the "problem out of the box" from Day 11.

n	1 mod 4 factors	3 mod 4 factors
1		
3		
5		
9		
15		
25		
45		
63		
65		
81		

Don't forget that 1 and 81 are factors of 81.

17. For each of the integers below, make a list of all of its factors that are congruent to 1 mod 3 and 2 mod 3. Look for patterns that relate to Part 2 of today's "problem out of the box."

n	1 mod 3 factors	2 mod 3 factors
1		
2		
4		
7		
8		
16		
20		
28		
40		
49		
64		
91		

Tough Stuff

18. Find three lattice points in $\mathbb{R}^2$ that are integer distances apart from one another, with none of the segments connecting them being horizontal or vertical.

19. Find a way to use any four consecutive Fibonacci numbers to generate Pythagorean triples.

20. There's a point inside most triangles that forms three 120° angles with segments to the three vertices. A Matsuura triangle is a triangle whose side lengths are all integers, and whose three interior segment lengths from the 120° point to the vertices are also integers. Find some Matsuura triangles, or prove they do not exist.

21. Little squares are cut out of the corners of a $m \times n$ rectangle and the whole thing is folded up to make a box of maximum volume. What choices of m and n will give good dimensions for the box?

22. Sketch a graph of this equation:

$$y = x^3 + 3x^2 - 144x + 140.$$

Find the x-intercepts and other interesting points. Find a way to produce a different cubic with integer roots, integer critical points, and integer inflection points.