13 *Baby by induction*

Important Stuff

PROBLEM

The following chart only contains N-values of Gaussian integers. So, 3 is missing because 3 cannot be the N-value of any Gaussian integer. Use the Sieve of the Other Sergio on this chart of N-values.

- Cross out 1 because Sergio says so.
- Circle 2, then cross out every multiple of 2.
- Circle the smallest number that hasn't been circled or crossed out yet (after circling 2 you would circle 5), then cross out all of its multiples. Some multiples may have already been crossed out—this is okay.
- Repeat the previous step until every number is either circled or crossed out.

1	2	4	5	8	9	10	13	16	17
18	20	25	26	29	32	34	36	37	40
41	45	49	50	52	53	58	61	64	65
68	72	73	74	80	81	82	85	89	90
97	98	100	101	104	106	109	113	116	117
121	122	125	128	130	136	137	144	145	146
148	149	153	157	160	162	164	169	170	173
178	180	181	185	193	194	196	197	200	202
205	208	212	218	221	225	226	229	232	233
234	241	242	244	245	250	256	257	260	261

Today's problem in the box fits in a box. Yay!

Which Sergio? We're not telling.

At what point can you stop this procedure and circle all remaining N-values?

1. Find all Gaussian integers $x + yi$ with $N(x + yi) = 49$.

Part 1 of the PootB from Day 11 might be helpful here.

2. Suppose $x + yi$ is a Gaussian integer with $N(x + yi) = 65$. If $a+bi$ is a factor of $x+yi$, what are its possible N-values?

3. (a) Suppose $x+yi$ is a Gaussian integer with $N(x+yi) = 49$. If $a + bi$ is a factor of $x + yi$, what are its possible N-values?

(b) Are you sure about that 7?

4. If $x + yi$ is a Gaussian integer with (real) prime $N(x + yi)$, explain why $x + yi$ can't be factored.

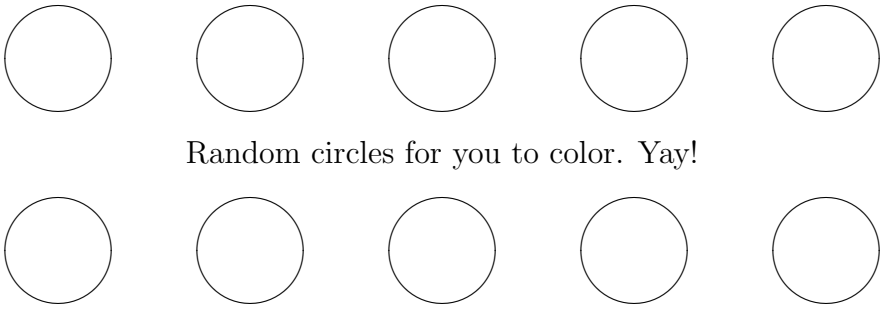
No boring factors like ± 1 or $\pm i$ are allowed.

5. Use the fact that $N(ab) = N(a) \cdot N(b)$ to explain why any Gaussian integer that has an N-value matching one of your circled numbers above must be a Gaussian prime.

A *Gaussian prime* is a Gaussian integer that is prime in i-land. In other words, all of the factorizations of a Gaussian prime involve boring factors like ± 1 and $\pm i$.

6. (a) Look at Part 1 of the "problem out of the box" from Day 11. How many circles contain the numbers between (and including) 1 and 36?

(b) Look at Part 2 of the "problem out of the box" from Day 11. Add up your numbers in the first three rows.



Random circles for you to color. Yay!

Neat Stuff

7. Prove that squaring Gaussian primes $a+bi$ with $a > b > 0$ gives primitive Pythagorean triples.

If you get stuck, ask Captain Caveman, a.k.a. Ben.

8. Get another copy of the grid of circles from Part 1 of the Day 11 "problem out of the box." (You probably want the one with the numbers filled in already.) Color all of the numbers that were circled in today's problem in the box.

9. The following chart only contains N-values of Eisenstein integers. So, 2 is missing because 2 cannot be the N-value of any Eisenstein integer. Use the Sieve of Sergio's Evil Twin Sergio on this chart of N-values. What's the significance of the numbers that end up getting circled?

Eisenstein integers, a.k.a. $w$integers, are numbers like $a + bw$ where a and b are integers. What would an old McDonald integer look like?

1	3	4	7	9	12	13	16	19	21
25	27	28	31	36	37	39	43	48	49
52	57	61	63	64	67	73	75	76	79
81	84	91	93	97	100	103	108	109	111
112	117	121	124	127	129	133	139	144	147
148	151	156	157	163	169	171	172	175	181
183	189	192	193	196	199	201	208	211	217
219	223	225	228	229	237	241	243	244	247
252	256	259	268	271	273	277	279	283	289
291	292	300	301	304	307	309	313	316	324

10. (a) Remember our favorite number w? Suppose $x + yw$ is an Eisenstein integer with $N(x+yw) = 49$. If $a+bw$ is a factor of $x + yw$, what are its possible N-values?
(b) Are you sure about that 7?

11. Describe a simple test to characterize if a Gaussian integer is prime.

12. Instead of using a sieve on the norms of Gaussian integers, one can use a sieve on the Gaussian integers themselves to find Gaussian primes. Using a piece of graph paper, adapt the Sieve of I-Can't-Believe-It's-Not-Sergio to find Gaussian primes.

You might want to start with a complex plane that goes from -7 to 7, $-7i$ to $7i$. Revisit at Problem 5 from Day 12 to get an easy way to locate the multiples of a Gaussian integer on your plane.

13. Get another copy of the grid of circles from Part 1 of the Day 12 "problem out of the box." (You probably want the one with the numbers filled in already.) Color all of the numbers that were circled in Problem 9.

14. Develop a definition for "Eisenstein prime."

15. Describe a simple test to characterize if an Eisenstein integer is prime.

16. For each of the following integers, count the number of its factors that have a remainder of 1 after dividing by 4, and the number of factors that have a remainder of 3 after dividing by 4. Write these numbers in the table below. Use Part 2 of the "problem out of the box" from Day 11 to complete the last row.

n	1	3	5	9	15	25	81	63	65	45	729	105
# of 1 mod 4 factors												
# of 3 mod 4 factors												
# appearances in grid											4	0

What is the connection between the rows of the table above?

17. For each of the following integers, count the number of its factors that have a remainder of 1 after dividing by 3, and the number of factors that have a remainder of 2 after dividing by 3. Write these numbers in the table below. Use Part 2 of the "problem out of the box" from Day 12 to complete the last row.

n	1	2	7	4	8	49	16	20	91	28	64	40
# of 1 mod 3 factors												
# of 2 mod 3 factors												
# appearances in grid												

What is the connection between the rows of the table above?

18. Find a way to generate all of the Pythagorean triples in which the hypotenuse and longer leg length are one away from each other.

Tough Stuff

19. Prove that every positive integer cannot have more factors that are congruent to 3 mod 4 than 1 mod 4.

20. Prove that every positive integer cannot have more factors that are congruent to 2 mod 3 than 1 mod 3.

21. Find a way to generate all of the Pythagorean triples in which the leg lengths are one away from each other.