15 *wow! that's BiG!*

Important Stuff

PROBLEM

Review all of the problem sets and make note of what you learned and thought about each day. Start with Day 1.

Celebrate what you learned. Don't you think this stuff is cool?!?! Did you have fun? What was the most interesting thing you came across?

You have to go back through all the previous problem sets anyway to do problem 0.

Supremely Unimportant Stuff

0. Let $x(n)$ be the height of the day number on the first page of the problem set for Day n. Is $x(n)$ linear, quadratic, exponential, or something else?

STOP! You're not allowed to go on until you've done the problem in the box.

Your Questions

Here's your chance to explore any of the mathematical ideas that have come up or any of the questions that you didn't have time to finish over the last three weeks. Have fun today! In case you can't think of something to investigate, here are some questions that you raised over the last three weeks.

Week 1 Questions

1. Why can't 103 be written as $N(x + yi)$?

2. What types of numbers give Pythagorean triples?

3. Which N-values can be produced by multiple positive pairs of x, y?

4. One hypotenuse. Two hypote......?

5. Is there a method that produces all Pythagorean triples? How do you know if you have all of the Pythagorean triples?

6. Do N-values have a geometric representation?

7. "Dudette, where's my $\sqrt{2}$?"
Translation: What is going on with numbers of the form $a + b\sqrt{2}$?

8. "Dude, where's my isometric paper?"
Probable Translation: What is going on with triangles that have a 60° angle and integer sides?

This question came up before we knew about Eisenstein triples.

Week 2 Questions

1. What does it mean when a number has N-value of 1?

2. What's up with rotations with $\sqrt{2}$?

What's up with this question?

3. Why do numbers of the form $a+b\sqrt{n}$ with $N(a+b\sqrt{n}) > 1$ seem to still appoximate $\sqrt{n}$ (using the method from Day 10)? Why is it slower?

4. Are there other types of dot papers?

5. Why do powers of i go by mod 4? Why do powers of w go by mod 3? How does this relate to the numbers we can make using $x^2 + y^2$ and $x^2 + y^2 - xy$?

6. What's going on with circles on square dot paper? Isometric dot paper? Same thing for ellipses and hyperbolas.

7. Andy has been feeling sad that 120° triangles have been neglected in favor of 60° triangles. Console him by defining $v = \frac{1}{2} + \frac{\sqrt{3}}{2}i$, calculating the distance from the origin to $[\rightarrow x, \nearrow y]$, then determining a formula for $N(x + yv)$ in terms of x and y, etc. Basically, do all the w stuff over again with v instead. How do things change in v-land?

If we find new triples, do we get to call them *Shawstein triples*?

Week 3 Questions

1. How can you test if a number can be written as a sum of squares?

Hmmm... look back at problems 16 and 17 on Day 13.

2. Is there an easy test to see if a Gaussian integer is prime?

3. Circles, ellipses, but no hyperbolas and parabolas?

Maybe it's time to revisit $\sqrt{2}$-land.

4. Can you convert a Pythagorean triple into an Eisenstein triple?

5. The average number of times that the numbers from 1 to 96 appear on the isometric grid of circles seems to be around 3.6. What's significant about that number?

Look at problem 14 on Day 7, and problems 7–9 on Day 14.

What? You Want More Math?!

Wow, you must really love math to be reading this far on the problem set. OK, here are some other things that we didn't get to explore.

Let $t(n)$ be the number of times that a positive integer n can be written as $x^2 + y^2$, where x and y are integers. The function $t(n)$ corresponds to the numbers you wrote in Part 2 of the "problem out of the box" for Day 11.

91. Use these values of $t(n)$ to figure out a rule for how to calculate $t(n)$. The values for $t(1)$ through $t(108)$ are on your chart. It might help to keep track of the prime factorization of n in this form:

$$n = p_1^{k_1}\, p_2^{k_2} \cdots p_m^{k_m}$$

where $p_1, p_2, \ldots, p_m$ are prime numbers.
(l) $t(5)$, $t(25)$, $t(125) = 16$, $t(625) = 20$
(i) $t(5)$, $t(10)$, $t(20)$
(s) $t(3)$, $t(9)$, $t(27)$, $t(81)$
(a) $t(11)$, $t(121) = 4$, $t(1331) = 0$, $t(14641) = 4$

(c) $t(5)$, $t(15)$, $t(45)$, $t(15)$, $t(30)$, $t(60)$
(o) $t(45)$, $t(90)$, $t(180) = 8$
(v) $t(13)$, $t(65)$, $t(845) = 24$
(e) $t(11)$, $t(55)$, $t(275) = 0$
(r) $t(130) = 16$, $t(260) = 16$, $t(520) = 16$

92. Once you think you've got a rule for calculating $t(n)$, check it against these values.
(v) $t(650) = 24$ and $650 = 2 \cdot 5^2 \cdot 13$
(i) $t(9945) = 32$ and $9945 = 3^2 \cdot 5 \cdot 13 \cdot 17$
(c) $t(19000) = 0$ and $19000 = 2^3 \cdot 5^3 \cdot 19$
(k) $t(488072) = 12$ and $488072 = 2^3 \cdot 13^2 \cdot 19^2$
(i) $t(510510) = 0$ and $510510 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17$

93. Once you've figured out the rule for $t(n)$, try $t_w(n)$, the function that counts the number of ways that n can be written as $x^2 + y^2 - xy$ with x, y integers.

The $t_w(n)$ function corresponds to the numbers you write in Part 2 of the PootB for Day 12.

94. Do problems 16 and 17 on Day 13 if you haven't already done them. Compare your formulations of $t(n)$ and $t_w(n)$ with what you find on those problems. Isn't it neat that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} t(n) = \pi?$$

We expect you to be jumping up and down at this point. Really.

95. Find the smallest number n such that $t(n) = t_w(n) = 24$.

96. Prove that no n exists with $t(n) = t_w(n) = 12$, even though there are infinitely many n with one or the other.

97. Find the smallest number n that is the third side in four primitive Pythagorean triples *and* four primitive Eisenstein triples.

Woo hoo hahahaha.

Totally Cool and yet Sadly Unrelated Stuff

98. Let $P_n(x)$ be a set of polynomials starting with $P_0(x) = 1$, $P_1(x) = x$. Generate the rest by the rule

$$P_n(x) = 2x \cdot P_{n-1}(x) - P_{n-2}(x).$$

Investigate these polynomials. What is their behavior? What roots do they have?

99. Trains, anyone?

100. Leibniz triangle, anyone?