Learning to Use Students' Mathematical Thinking to Orchestrate a Class Discussion

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Abstract

Learning to orchestrate class discussions that are based on students' mathematical thinking is one of the most difficult aspects of learning to teach in ways that build on students' mathematical experiences. Based on a research project in which student teaching was restructured so as to focus on using student thinking, we describe the steps of a process that teachers move through when using students' mathematical thinking. We also identify some roadblocks that keep student teachers from listening to and understanding, from recognizing, and from effectively using student mathematical thinking for classroom discussion. We discuss how understanding this process and these roadblocks can be useful to mathematics teacher educators in their work with preservice mathematics teachers.

Keywords: Orchestrating Classroom Discourse, Preservice Teacher Education, Secondary Student Teaching, Student Mathematical Thinking, Teacher Knowledge, Teaching Practice

Learning to Use Students' Mathematical Thinking to Orchestrate a Class Discussion

Mathematics classrooms wherein the teacher promotes mathematical discussion based on students' mathematical thinking[1], and then orchestrates that discussion in ways that facilitate yet deeper mathematical thinking, provide opportunities for students to meaningfully "struggle with important mathematics," (Hiebert & Grouws, 2007, p. 387) something that research has shown is critical for students to learn with understanding (Hiebert & Grouws, 2007; Hiebert et al., 1997; National Council of Teachers of Mathematics, 1991, 2007). Orchestrating such discussions, however, seems to be one of the most difficult aspects of this approach to teaching (Sherin, 2002a), particularly for novice teachers. This paper reports on the results of a research project that tried to put student teachers (STs) in situations where they were trying to elicit students' mathematical thinking, and then studied how they navigated the road to using that thinking in their teaching.

USING STUDENTS' MATHEMATICAL THINKING

The NCTM (2007) recommended that mathematics teachers "orchestrate discourse by… listening carefully to students' ideas and deciding what to pursue in depth from among the ideas that students generate during a discussion" (p. 45). It is this careful listening to, and pursuit of students' ideas that we refer to as *using students' mathematical thinking*; we refer to such opportunities to use students' mathematical thinking as *teachable moments*. Although *teachable moment* is not a well-defined construct in the literature, the idea of teachers capitalizing on students' mathematical thinking "in the moment" is frequently discussed in the literature on mathematics classroom discourse (e.g., Doerr, 2006; Manouchehri & St. John, 2006; Schoenfeld, 2008). In this section we describe a conceptualization of the process of using students' mathematical thinking that both informed and evolved over the course of our study. In order to effectively use students' mathematical thinking, teachers need to 1) listen to and understand student thinking, 2) recognize the thinking as a teachable moment, and 3) use the thinking for a mathematical and pedagogical purpose.

Listening and Understanding

Effective teaching "requires careful listening" (Erickson, 2003, p. ix), in large measure because effective teaching builds on what and how students (particularly those present) think. In order to use students' mathematical thinking, teachers need to listen with the intent of using that thinking in order to build the classroom understanding of the mathematics at hand. Such listening must occur both when teachers explicitly elicit their students' ways of thinking as well as in the myriad moments that arise unexpectedly. Although careful listening creates teachable moments that serve meaningful purposes beyond content (Schultz, 2003), the listening to which we refer in this paper is content-specific. We are speaking of listening to students' *mathematical* thinking with the intent to use that thinking in order to further the learning of *mathematics* for all students in the classroom.

From our countless daily interactions, each of us can attest to the fact that it is possible to listen yet not understand. Thus, teachers may listen to students explain their thinking but may not understand that thinking. In order to understand students' mathematical thinking, teachers themselves must have an understanding of the mathematical concepts at hand (Ball, Lubienski, & Mewborn, 2001; Ma, 1999; Sherin, 2002b). Although we find it valuable to consider *listening* and *understanding* as separate steps in the process of using students' mathematical thinking, it is often difficult to distinguish between the two in practice.

Listening to students has long been both advocated and studied by educators (e.g., Confrey, 1993; Davis, 1997; Paley, 1986; Schultz, 2003). Davis, for example, considered

[1] By "students' mathematical thinking" we mean students', among other things, their solution strategies, their justifications and reasoning, and their models and representations.

different types of listening that led to different types of teacher actions. Listening was characterized as *evaluative* when it was somewhat superficial and as *interpretive* when it sought merely to understand; with both types of listening there was no apparent intention by the teacher that the results of listening and understanding students' thinking would inform or redirect the lesson. Such listening could be classified as *funneling* (Wood, 1998), wherein a teacher listens for student thinking that will lead toward a preconceived "best solution" and away from alternative and wrong strategies.

In contrast to evaluative and interpretive listening, Davis' (1997) *hermeneutic* listening had at its very core the notion that students' thinking would in large measure determine the direction of the lesson. Wood's (1998) *focusing* pattern encompasses Davis' (1997) hermeneutic listening as well as other types of pedagogically-sound listening. In the focusing pattern, the teacher listens for alternative and incorrect strategies as a means of elevating (and focusing) students' mathematical thinking toward important mathematical ideas. We thus adopt *focused* listening to describe the listening in our process—listening with the intent to understand and then to meaningfully use students' mathematical thinking in order to further mathematical objectives.

Recognizing the Teachable Moment

Once a teacher has listened to and understood a student's mathematical thinking, he must then *recognize* this moment as a teachable moment; he must recognize the potential pedagogical and mathematical value in pursuing that thinking in order to be able to eventually use it. Although good intentions and common content knowledge are likely sufficient to allow a teacher to listen to and understand students' thinking, recognizing such thinking as a teachable moment takes a great deal of specialized knowledge. Recognizing such moments may also depend on the goal of the lesson, unit or course and may depend on how the student thinking that was shared fits with the flow of the lesson. Lewis & Tsuchida (1998) quoted a Japanese teacher as saying, "A lesson is like a swiftly flowing river; when you're teaching you must make judgments instantly" (p. 15). Recognizing shared student thinking as a teachable moment is one of the instantaneous judgments that are made during lessons. Recognizing the pedagogical and mathematical value in students' thinking in the moment is a difficult step in the process of using students' thinking, even for experienced teachers (Chamberlin, 2005).

Using Students' Mathematical Thinking Effectively

Once a teacher has listened in a focused way to students' mathematical thinking, has understood that thinking, and has recognized the mathematical and pedagogical value of that thinking, he is in a position to use that thinking. In our conceptualization, the purpose of such use is to help all students to gain a better understanding of the concept at hand. Thus the thinking that is used may be correct or incorrect and this use is more than just sharing different methods for solving the same problem. Effective use of students' mathematical thinking requires the teacher to orchestrate a discussion about the connections between the different methods or discuss why some methods work and others do not. Effective use involves more than explanations of the methods or thinking; it involves making explicit the reasoning behind the thinking.

The Influence of Teacher Knowledge on the Process

We have found Hill, Ball & Schilling's (2008) conceptualization of teachers' knowledge to be a useful way to think about our process conceptualization—how one listens to, understands, recognizes as valuable and uses students' mathematical thinking. In their conceptualization, Hill et al. consider subcategories of Shulman's (1986) content knowledge and pedagogical content knowledge. These subcategories begin to delineate some of the specialized knowledge that teachers have and that others do not. Thus, although mathematics teachers have mathematical knowledge shared with those who are not teachers (common content knowledge – CCK), teachers also have specialized mathematical knowledge; this knowledge, such as knowledge of the affordances and constraints of various mathematical representations and models, is not commonly held by the general public or even by working mathematicians, but is integral to the work of a mathematics teacher (specialized content knowledge – SCK). Hill et al. (2008) also consider different types of pedagogical content knowledge: knowledge of content and students (KCS), knowledge of content and teaching (KCT), and knowledge of curriculum.

We now consider how these types of knowledge aid us in viewing the process of using students' mathematical thinking. For example, in reflecting on our own experience as mathematics teachers, a common scenario often gives rise to teachable moments—students often make mathematical statements that are slightly incomplete or incorrect. Various types of mathematical knowledge for teaching help us to take advantage of such teachable moments. Our KCS might prompt us to listen very closely to the response to a particular question because we know that students often have misconceptions in this area and that an incomplete answer might be masking just such a misconception. Our knowledge of curriculum might help us to recognize that the incompleteness of this response is likely connected to a bit of mathematics that is just on the mathematical horizon (Ball, 1993), and thus one that would be valuable to pursue. Our specialized content knowledge (SCK) might help us use the incompleteness of this response as just the right motivation to discuss a different representation or model for the mathematics at hand. We thus view mathematical knowledge for teaching as critical for teachers to be able to carry out the process of effectively using students' mathematical thinking.

Illustrating the Process Conceptualization

In order to illustrate the process of using students' thinking that we have just described, we present here an episode from our data wherein a ST uses her students' mathematical thinking quite effectively. We then point to the evidence within the episode and from the STs' reflections that allow us to use the *process conceptualization* in order to interpret the episode.

ST Emily[2] began her pre-algebra lesson by having students work individually for several minutes on three true-false questions regarding similarity of geometric figures. She then began a class discussion by asking for volunteers to share their decision on the first statement: "All squares are similar." Christopher volunteered and stated that he thought that the statement was true because, in a given square, all of the sides are equal, all of the angles are equal, and opposite sides are parallel. ST Emily followed up briefly and then asked whether anyone thought the statement was false. Brandon volunteered, but also argued that the statement was true, although for a different reason:

Brandon: I think it's true because, no matter what, they both have parallel lines, and if you just draw two sets of parallel lines you get a square.

ST Emily: Okay. What do parallel lines tell you about them being similar?

Brandon: You have to have two sets of parallel lines to be a square, so obviously, if it's a square they all have parallel lines so that's what's similar.

ST Emily: So, if something has parallel lines, that makes it similar?

Brandon: Well, to be a square they have to have parallel lines. If they're a square, they all have parallel lines. So if you have like four squares, they're all similar because they all have two sets of parallel lines.

[2] All names for STs and students are pseudonyms.

By this point in the conversation there were many students in class who were shaking their heads, saying "no, that's not right," and raising their hands in hopes of responding to Brandon's thinking.

> ST Emily: Okay. What do you think about that Kayla?
> Kayla: I think that's not true—it is true, but—.
> ST Emily: Okay. What's not true and what is true?

Kayla then went to the board, drew two squares (one with side length 1 unit and one with side length 2 units), and then argued that there was a common scale factor of 2 between the two squares. Although it is never stated explicitly, apparently Kayla agreed with Brandon about the pairs of parallel lines always existing in squares, but not in his use of this reasoning to conclude that all squares are similar. Samantha then asked Kayla why she found the scale factor as part of her explanation, to which Kayla responded, "Because that's what we found for similarity last time." Samantha then continued the conversation:

> Samantha: You could also kind of verify with a square and a rectangle. Rectangles, they have to have parallel sides, but squares, all their side lengths have to be the same.
> ST Emily: Okay. So how is that different from what Brandon was saying?
> Samantha: Brandon was saying that they just had to be parallel.
> ST Emily: So can you show us a rectangle that's not similar—that has parallel sides?

Samantha went to the board, drew a skinny rectangle and a square, and pointed out that each figure has pairs of parallel sides. Then ST Emily asked the class whether the square and the rectangle are similar:

> Terry: Yeah, they have parallel lines. Because that one on top is the same as that [apparently comparing the top and bottom sides of the square].
> ST Emily: Okay, so we know that they are parallel, but does that make them similar? [Various students say "No" and "Maybe."] What do we have to know in order for something to be similar? [Numerous students, including Brandon, start to share their responses, some mentioning scale factor.] Let's listen to Brandon for a second. What did you say?
> Brandon: Similar means they have something in common. The thing that they have in common is parallel sides. [Lots of murmuring amongst the students]
> ST Emily: It's true that when we're talking about something being similar they have to have something in common. But when we're talking about something being *mathematically* similar—
> Alex: It has to— You have to find the scale factor.
> Cody: Doesn't it have to have the same angles and the same sides?

ST Emily then orchestrated a discussion with Cody and others about the two criteria for figures to be mathematically similar (i.e., scale factor and equal angles). Having done this, ST Emily turned the conversation back to Brandon.

> ST Emily: So, Brandon, is it enough for the lines to be parallel for it to be similar?
> Brandon: No.
> ST Emily: No. Why not?
> Brandon: Because they have to have corresponding sides and then corresponding lengths and stuff like that. Just parallel sides wouldn't be enough.
> ST Emily: Okay. Do we all understand that? It's true that they do have something in common, Brandon, but they're not mathematically similar.

We now analyze this episode according to the process of using students' mathematical thinking. We find evidence that ST Emily listened to Brandon's initial contribution in her first follow-up question, which also represents ST Emily's attempt to better understand Brandon's thinking. Her second question demonstrates that she understood his claim. She later reflected on the beginning of this conversation in this way: "I continued to push him to explain until I understood that his definition of similar was having something in common." At least in part, it was ST Emily's KCS that contributed to her ability to listen to and understand Brandon's thinking.

That ST Emily recognized this situation as a teachable moment and felt as though she had taken advantage of it was also revealed in her reflection: "Then I was able to validate his thinking and talk about what it means to be mathematically similar." We find this statement to be somewhat understated. ST Emily allowed the class to respond to Brandon's thinking and that thinking did not always directly address the underlying issue Brandon had raised. ST Emily brought the conversation back to Brandon's thinking on several occasions, thus helping to focus Brandon and the rest of the class on the distinction between the common and the mathematical definitions of *similar*. Her SCK provided her the ability to draw this distinction and her KCT helped her to direct the class discussion in that direction. This episode thus demonstrates the process of listening to and understanding, recognizing, and effectively using students' mathematical thinking and the role that mathematical knowledge for teaching played in supporting ST Emily in carrying out that process.

METHODOLOGY

This study took place in the context of a larger project wherein we altered the structure and purpose of student teaching in an attempt to emphasize the elicitation and use of students' mathematical thinking while deemphasizing survival and classroom management. In this student teaching project, a pair of STs was placed with one cooperating teacher and two or three pairs of STs at different schools formed a cluster. The STs taught at most one lesson per week during the first 5 weeks of their 15-week student teaching internship. These lessons were planned in their pairs but were taught individually and observed by the other STs in the cluster, the cooperating teacher and the university supervisor. Following the teaching of the lessons, a reflection meeting was held in which the STs who taught the lesson would reflect on the goals of their lesson and on how the tasks of the lesson were intended to meet those goals. The observers then had an opportunity to ask questions and to make comments. As part of the altered structure, the STs also conducted directed observations and student interviews, and wrote weekly reflection papers (all focused on students' mathematical thinking) as a means of processing and synthesizing what they were learning.

For this study six female STs (Emily, Christina, Holly, Megan, Jennifer and Ashley) and three cooperating teachers were purposefully selected to participate. The STs were chosen based on feedback from their past professors, who were asked to recommend students who they felt were primed to excel during student teaching. Emily and Christina were placed in a middle school and Holly and Megan were placed in a junior high school. These four STs were placed with teachers who were approaching their instruction from an NCTM *Standards* perspective and were using a reform-based curriculum. Jennifer and Ashley were placed in a high school setting with a new teacher who taught fairly traditionally but was open to learning new ideas and who supported the STs in implementing such ideas.

The data for this study consisted of video recordings of all ST lessons and the accompanying reflection meetings as well as the reflection papers that the STs wrote regarding these observations and reflection meetings. To identify candidates for teachable moments, the reflection meetings were analyzed for specific comments made by the person who taught the lesson or by observers. The comments of interest were those that made reference to student thinking observed in the lesson. Once these comments were identified, the lesson was analyzed to locate the episode that was being referenced. The episodes were then analyzed to determine whether they were teachable moments. This determination was based on whether both researchers felt that they might have pursued the students' mathematical thinking had they been teaching the class. In addition, the reflection papers written by the STs who taught the lessons were analyzed to identify any additional thoughts they had on the identified episode.

Once these teachable moments, as well as any comments or reflections about them, were identified an analysis of how the teacher used the student thinking ensued. In that analysis we viewed the episode through the lens of our conceptualization of the process of using students' mathematical thinking, seeking evidence as to whether each step was accomplished by the ST. Having identified the stopping points in the process, we looked for evidence of why the ST stopped the process where she did. This assessment was done by evaluating comments made during the lesson, during the reflection meeting or in the reflection paper. From these various data sources, we identified and describe here a variety of roadblocks that inhibited these STs from effectively completing the process of using student mathematical thinking.

ROADBLOCKS TO EFFECTIVE USE OF STUDENTS' MATHEMATICAL THINKING

All the STs believed to some extent that their lessons would be more productive if their students were given opportunities to make comments or to share their solutions to problems. Therefore, there were many times during their lessons when they elicited students' mathematical thinking; often these instances could be viewed as teachable moments. As was expected of novice teachers, and regardless of whether the STs were using reform curricula in a middle school or traditional curricula in a high school, they ran into similar issues when trying to conduct a whole class discussion that used their students' thinking in order to assist all students to come to a deeper understanding of the underlying mathematics. We describe here a collection of roadblocks that hindered the process of effectively using student thinking for classroom discussion. Although we use the term "roadblocks," we view such instances in a positive light. As mathematics teacher educators we were pleased to see our STs grappling with these important issues—bumping up against important dilemmas of teaching. The identification of these roadblocks informed us as to where we needed to go as teacher educators in our efforts to help the STs continue their development as mathematics teachers.

Roadblocks to Listening and Understanding

The literature is replete with examples of novice teachers (e.g., Borko et al., 1992; Cooney, 1985; Schultz, 2003) and experienced teachers (e.g., Ball, 1993; Davis, 1997; Lampert, 1990; Schultz, 2003) who struggle to attend to the complexities of teaching. It is no small task for teachers to balance attending to what students are saying with attending to what they will do or say next. Thus, one major roadblock to listening seemed to be the inability to attend to student thinking *and* attend to other aspects of teaching. In addition, even when the STs were listening to the substance of their students' thinking, they sometimes struggled to *understand* what was being said. When leading a class discussion where students are encouraged to share their thinking and methods for solving a problem, a ST's knowledge or experience may not allow her to understand a student's thinking. Because the student strategy is unique or different, the ST may not understand the point the student is trying to make, even though she is listening. Student thinking that is not understood cannot be used to enrich the class discussion for the benefit of all students.

An example of a roadblock to listening and understanding occurred as ST Jennifer taught a pre-calculus lesson that she had planned with her partner ST Ashley. The task that they had created was a set of cards containing different linear functions represented using words, graphs or equations. The students were asked to classify the cards according to their attributes, such as increasing or decreasing slopes. The STs wanted the students to reflect on the attributes of parallel, perpendicular, vertical and horizontal lines by looking at the similarities and differences among the various representations of a line. However, the STs had created this task by adapting an activity they had done in a university class, where each of several different kinds of functions was represented in four different ways – numerically, graphically, verbally and algebraically (see Cooney, Brown, Dossey, Schrage, & Wittmann, 1996, pp. 41-45). One of the primary purposes of this original task was to help preservice teachers review many different types of functions while simultaneously considering the attributes of these multiple representations. The STs adapted the original task to have three different representations of each of several linear functions that varied according to their slope or y-intercept. Because this "matching" characteristic of the original task still existed, many of these pre-calculus students attempted to group their cards only according to the three different representations of the same function, rather than considering classifications based on more general characteristics such as increasing or decreasing slopes.

Although much of the expressed student thinking was focused on grouping the cards according to the different representations of the same function, some of the thinking was clearly related to slope, one of the intended foci of the activity. However, as ST Jennifer elicited her students' thinking as part of a class discussion, she mostly just commented, "Okay, okay. Yeah, that's interesting." and then moved on. A similar behavior was seen as ST Jennifer moved from group to group prior to the class discussion. She asked one group how they were classifying their cards but did not ask any follow up questions about what they were looking for or why. After looking at another group's work, she said "Oh, [you are classifying] by y-intercept. Good job." This student thinking seemed to meet the lesson goal and yet she had no further discussion beyond this comment. Thus, ST Jennifer was not listening to her students' thinking, even though some of it could have helped her to meet her mathematical goals.

In the reflection meeting, ST Jennifer was asked to explain the classification she was hoping to see. She responded by saying,

> We didn't really expect them to say, "Okay, well, this is the graph, it matches this equation, it matches this story." We didn't think they'd do that right off the bat…. Every single group did that…. The first thing that they came up with every time is, "We just matched them up."

Throughout this discussion ST Jennifer continually returned to the problematic nature of the unanticipated responses. We believe that she was so preoccupied with the number of unanticipated solutions that she did not listen for the thinking that might lead to her goal. In this case, ST Jennifer's attention to the seeming failure of her task hindered her from listening to her students' thinking.

Concerns about classroom management also functioned as roadblocks to listening and understanding. With respect to this same card-sorting lesson, ST Jennifer said the following in her reflection paper:

> I was noticing that some students were finished with the activity, so I asked them to write their answers on the board to give them something to do. Had I been more aware of their answers, I would not have had them present. Their answers were not beneficial to the class discussion.

In this case, ST Jennifer did not listen to her students' thinking before she attempted to *use* it. Her attention to issues of classroom management hindered her from listening to the student thinking that she was observing in the class.

In summary, two of the roadblocks to listening and understanding are the challenge of keeping the classroom running smoothly when the lesson feels like it is falling apart and using student thinking for a management purpose instead of using it to better understand the mathematics. As mentioned previously, effective teaching is a complex endeavor. Learning to use students' mathematical thinking requires learning to attend to that thinking while attending to many other aspects of the class and of the lesson.

Roadblocks to Recognizing the Teachable Moment

It is one thing to understand what a student says. It is quite another thing to recognize that thinking as a teachable moment—to understand the significance of what the student has said and to see value in that thinking from a mathematical and pedagogical stance. We have identified a number of roadblocks to such recognition.

Assumption of Understanding

STs often work on the assumption that their students already understand the mathematics at hand. Our data revealed two variations of how these assumptions play out.

Fill in the blanks. Novice teachers have a tendency to implicitly "fill in the blanks" when their students are talking about mathematics rather than asking the students to do so (cf. Ball, 2001). Students often use imprecise language when answering questions or sharing their work. The STs frequently forgave this imprecision, assuming that the student understood what they were superficially or inadequately describing. They failed to recognize such moments as important opportunities to push the student to clarify their statements and thinking.

An example of this roadblock occurred in a lesson taught by ST Jennifer. One of the homework questions had asked the students to find a line through the point (6, 5) that is perpendicular to $y = -\frac{2}{3}x + 2$. As a class the students had arrived at the equation $y = \frac{3}{2}x + b$. ST Jennifer then asked what they needed to do next for the line to pass through (6, 5). ST Jennifer described the student response and her thinking as follows:

> I got the chorus answer "you plug in (6,5)." I then assumed that most students knew how to do this and moved on. After the reflection meeting, I now see this situation differently. If I was in this situation again, I would ask why I can't plug in any value I want to. This discussion probably could have deepened students' understanding of an equation for a line. I hope that next time, I can be more aware of little situations like this that could strike up a mathematically engaging discussion.

It is clear from ST Jennifer's comments that she assumed the students understood the underlying mathematics when they said "you plug in (6, 5)" and filled in the blank about why plugging in (6, 5) yields the desired results.

Simply remind. When students display incorrect or incomplete thinking about mathematics that has been recently talked about in class or that they have learned in the past or that was written in the instructions, the STs tended to assume that the students actually understand it and that they "just need to be reminded" about it (thus equating learning with memorizing). Such incomplete or incorrect student thinking was often viewed by the STs as "mistakes" rather than "misunderstandings." In these instances the STs tended not to question students' understanding of that mathematics. Instead they either reminded the students of the time or place where the concept had been addressed before or rephrased the student comments by correcting or completing the response.

An example of this roadblock occurred when ST Holly was teaching a lesson wherein the students were asked to complete a table describing the distance from a motion detector at time t as a person walks toward it. (Although this task was carried out without a motion detector, the students had been involved in an activity where they had used the motion detector earlier in the week.) The students were given tables with several values of t already included and were asked to complete the tables and to graph their results. Some of these values of t could be interpreted to mean that the person had walked past the motion detector.

The table was labeled with time as the independent variable and distance as the dependent variable. The STs wanted the students to enter negative distances once the person walked past the motion detector even though this solution was not clear from the context. There were many students who did not use negative numbers in their solutions. As ST Holly interacted with those groups or students she gave a variety of little hints about how they might fill in the paper. In her reflection paper she commented that she had tried to resolve the issue that students were not using negative numbers in their solutions by encouraging "the students to read the problems carefully and to make sure what they were saying." She also said, "I know the students just didn't read the problem correctly." ST Holly did not see the situation as being problematic for the students and assumed that they would have understood if they had just read the instructions more carefully. Her tendency to view the students' alternate solutions as the result of mistakes rather than as attempts to make meaning of the task hindered ST Holly from recognizing this thinking as a teachable moment.

In each case here, the STs' assumption of understanding inhibited them from recognizing the moment as a teachable moment. They assumed that when a student provided a simple response that was correct, the student had the desired depth of understanding the STs were seeking. They also assumed that when a student made an incorrect statement, they had "just forgotten" but they really understood the concept at hand. Both of these types of assumptions kept the STs from recognizing the potential rich conversation that could occur if only they dug a little deeper.

Funneling

The other main roadblock to recognition that we identified in our data was referred to earlier in this paper: "funneling" (Wood, 1998), or looking for a particular response and, in the process, failing to recognize the mathematical and pedagogical significance of other responses. It may seem as though funneling could be categorized earlier in the process, as a roadblock to listening, but we do not think this is the case. In order to funnel, one must actually listen to student thinking and understand it enough to recognize that it is not the thinking being sought. Thus, STs who funnel have at least listened and understood. What they fail to do is to recognize the mathematical and pedagogical significance of the response. Because they have a preconceived notion of the response that will lead to a teachable moment, they fail to recognize other responses that may lead to similar or even different but still valuable teaching moments.

An example of funneling that prevented a ST from recognizing a teachable moment occurred when ST Christina was launching a lesson wherein students were going to input equations into a calculator and look at the tabular outputs to make decisions about the situation. In anticipation of this approach, ST Christina asked the students to identify the independent and dependent variables in the equations $A = \pi r^2$ and $C = 2\pi r$. This activity was meant to be a quick review so that the students would be able to input equations properly into the calculator. ST Christina wanted to hear that *r* was the independent variable and that *A* and *C* were the dependent variables in their respective equations. After a student had shared his response that *r* was the independent variable and *A* was the dependent variable in the former equation, Morgan said that she thought *C* was the independent variable and *r* was the dependent variable in the latter equation. Because this response was not what ST Christina wanted, she began a funneling process:

ST Christina: Morgan, come up and explain to us what you have here.

Morgan: I did the circumference because the radius depends on how big or small the circle is. So I said the circumference is independent and the radius is dependent on the circumference.

ST Christina: OK. Thanks Morgan. Who has the same thing as Morgan? Who has something different? Who doesn't know? [pause] Who said they have something different? Brian's the only one who has something different?

Sage: I don't know.

ST Christina: Nobody else? Katherine, do you? Do you want to explain?

Katherine: I just said the independent would be the radius and the dependent would be the circumference.

ST Christina: OK, Why?

Katherine: Because… the circumference is the—. Wait, no, I agree with her [Morgan].

ST Christina: Are you sure? You were going good there. Do you want to keep explaining what you were saying?

Katherine: I was going to say that the circumference would change if the circle gets smaller. But um, you can find the circumference without the radius I think.

ST Christina: You can find the circumference without the radius? How would you do that?

Katherine: Um. I don't know.

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ST Christina: Brian, what do you think?

Brian: Couldn't kind of both of them go both ways? Because like in area. Like as the area gets smaller so does the—. Oh, never mind.

ST Christina: So let's look back at this one. How did these equations relate with each other with the independent and dependent, um, with both of them and how can we think this through? Anybody have some ideas besides Brian? Brian, thanks for your help though. Abe, what do you think?

In this episode Morgan presented a solution that was not what ST Christina had anticipated, so she asked the class if someone had approached the situation differently and this is where the funneling began. Katherine started to say that she disagreed with the first student and then changed her mind. ST Christina tried to pursue Katherine's initial thinking because it was what she was looking for, namely r as the independent variable and C as the dependent variable. Brian then suggested that it could go either way and then backed off as did Katherine. In this case, however, the ST Christina did not pursue Brian's comment. With both Katherine and Brian ST Christina funneled toward her preconceived correct solution and away from the divergent thinking of the students. This funneling had ST Christina trying to get Katherine to explain her original thoughts because they supported her goal and yet the funneling hindered ST Christina from recognizing the richness of Ben's thinking as a teachable moment.

Roadblocks to Effectively Using Students' Mathematical Thinking

Because STs are in the process of learning how to teach, it comes as no surprise that they might listen to student thinking, understand what students have said, recognize that thinking as a teachable moment, yet still not be able to use the student thinking in a way that furthers their mathematical learning goals—that capitalizes on the teachable moment. Our analysis of the data revealed a number of roadblocks to effective use of student thinking. In our categorization of these roadblocks, we make a distinction between roadblocks to trying to use student thinking and roadblocks to effective use. The former roadblocks inhibited the STs from even attempting to use their students' thinking. With the latter roadblocks, the STs attempted to use their students' thinking but fell short.

No Attempt to Use

First, we located a number of instances in our data where the STs were able to recognize student thinking as a teachable moment and yet they made no attempt to use that thinking in their lesson (i.e., they did not pursue the thinking with the class). When such instances occurred we could trace the reason to a lack of knowledge, usually a lack of either SCK, PK or CK. We use several episodes to illustrate such roadblocks:

Lack of SCK. ST Holly had given her pre-algebra class the table shown in Figure 1, which gives the number of people out of 100 surveyed who would go on a bike tour for the given total prices. The students were asked the following question: "To make a graph of these data, which variable would you put on the *x*-axis? Which variable would you put on the *y*-axis? Explain." The students were also asked to "make a coordinate graph of the data on the grid paper" (Lappan, Fey, Fitzgerald, Friel, & Phillips, 2006, p. 32).

ST Holly had anticipated "that students would have problems with the independent and dependent variables," so during the first few minutes of the task she "went around to the different groups… looking for students that had their independent and dependent variables labeled correctly and also students that had their variables mixed up." Having noticed that many students seemed to be struggling a great deal with the identification of the independent and dependent variables, ST Holly decided to bring the class together for a class discussion. In order to initiate the class discussion she polled the class:

ST Holly: How many of you guys think that the *total price* is the independent? [Quite a few students raise their hands.] How many of you think that the *number of customers* is the independent? [Some (but fewer) students raise their hands.] Okay. Someone who thinks that the *number of customers* is independent, will you tell me why?

Spencer: I'd say that it's the *number of customers* because the customers depend on their opinion of what the price should be.

ST Holly: Okay. So the number of customers depends on what the price will be?

Spencer: No. The money depends on what the number of customers should be.

ST Holly: Okay. Is that what it says in the prompt? That's the one I did. Is there another idea why that one would be the independent one?

Breanna: You can't, because it's how many people want the price.

ST Holly: How many people want that price?

Breanna: Yeah. So, like, it's kind of hard to explain.

ST Holly: It's kind of hard to explain? Okay, someone who picked the *total price* to be the independent one, do you want to give me an explanation?

George: Yeah. I think it's the *total price* because, like, just because they want to go doesn't mean they can change the price, so the price stays the same. [ST Holly: Uhuh.]. So the number of people changes depending on if they want to pay that much or not.

ST Holly: Okay. Does everyone understand what he said? A little bit? Okay, let me give you another example.

ST Holly then proceeded to give an example that she and ST Megan had developed while planning their lesson, one that they had hoped would help students gain a better understanding of how the analysis of the context of a situation helps you to identify the independent and the dependent variables:

ST Holly: Ariel, let's say I come up to you and I'm like, "Do you want to buy my ipod?" What do you say?

Ariel: Um, no.

ST Holly: No? Why not? Why don't you want to buy my ipod?

Ariel: Well, probably because you may have used it already and everything and you already have your own songs that you already had on it.

ST Holly: Uhuh. That's a good thing. So it kind of depends on different things, right?

Ariel: Yeah.

ST Holly: So I can't just, like come up to you and be like, "Do you want to go on the bike tour?" You probably want to know how much it is first. Right?

Ariel: Uhuh.

ST Holly: So that's something to think about.

ST Holly then asked the students to get into pairs and return to the task.

In reflecting back on this class discussion ST Holly recognized that the discussion "never came to closure" on the issue of determining the independent and dependent variables. She then shared this important bit of insight into her thinking at the time:

> People would argue both sides and I could see both sides, but I didn't know how to justify them. And I just kept trying to bring it back to the context. But every time I'd bring it back to the context they'd come up with something different and I was like, "Oh, I didn't think of that." And so I really didn't know how to—. So I don't know.

In this episode ST Holly asked questions that allowed her students to share their thinking about determining the independent and dependent variables in this situation. There is evidence that she was listening to and understanding what they said (i.e., "I could see both sides"). She also recognized this discussion as a great opportunity to talk about the very thing she wanted to talk about—deciding which variable should be independent and which should be dependent. What seemed to keep ST Holly from using the students' thinking in this situation was a lack of SCK. ST Holly had developed a strong enough knowledge of independent and dependent variables to know that such classification was highly dependent on the context. She did not have sufficient understanding of the underlying mathematics, however, to allow her to see how to justify or refute the various responses that she received. Without this knowledge she was unable to push on that thinking, to point out what was correct and what was incorrect in the students' thinking. Rather than use the student thinking that had been proffered, her lack of SCK forced her to retreat from that thinking and to introduce her own thinking. In this episode ST Holly's lack of SCK was a roadblock to her using her students' mathematical thinking.

Lack of Pedagogical Knowledge. Lack of pedagogical knowledge (PK) also impeded the STs from using their students' mathematical thinking. One of the most common ways this lack of knowledge revealed itself was when the STs would notice a productive conversation within a small group. The STs would often listen to such conversations, recognize them as valuable, and have the desire to use that thinking in a class discussion. They soon learned, however, that recreating such individual group discussions was not easy. Their approach usually took the form of asking the group to share with the class the conversation they had just had; this approach never succeeded. As Ashley put it, "It was really fun to sit in on their argument, but for me, it's hard to recreate that argument in the classroom discussion because they feel like they've already had the conversation; they don't want to have the conversation again." In our experience, although students may indeed be reluctant to reproduce such conversations, such reproduction is practically impossible for them. Students tend to focus on the results of their conversations, seldom on the process or on pitfalls of those conversations. Thus, successful reproduction of valuable small-group conversations must be facilitated by the teacher, and usually entails involving the rest of the class in the issue that was being discussed, thus recreating the situation for the whole class that caused the productive conversation in the small group. This pedagogical knowledge of how to use students' mathematical thinking was not yet available to the STs in our study, and the lack of this knowledge was a roadblock to their use of that thinking. We note, however, the important progress the STs were making in that they were beginning to recognize their lack of this knowledge and to find ways to acquire it.

Lack of Curricular Knowledge. The STs' lack of curricular knowledge often impeded their ability to use students' mathematical thinking. For example, in the motion detector task described earlier, many students questioned the notion of negative numbers in the context of the problem, because "the students were unsure about the motion detector being able to read the person if they were behind the motion detector" (ST Holly). Students who focused on the context of the problem questioned the use of negative numbers and many started to develop solutions that were building toward the notion of absolute value. The STs recognized that the solutions were headed in that direction and chose not to pursue them. As ST Holly stated, "I stayed away from that idea because I didn't want to talk about absolute value functions. I didn't even know if it was ok to talk about absolute value functions this early." In this situation ST Holly's lack of curricular knowledge impeded her from using the student thinking that was elicited by this task. She did not have a sufficient understanding of the connections between the current day's mathematics topic and the underlying mathematical ideas of absolute value.

Thus, in general, the STs in this study were often inhibited from using student thinking because they did not understand the mathematics enough to pursue the thinking with their students (lack of SCK), they did not know how to carry out that pursuit (lack of PK), or they did not know whether they "should" pursue it (lack of curricular knowledge). In each case the STs had sufficient knowledge to listen to, understand and recognize the value in their students' thinking; what they they lacked was the knowledge to use that thinking.

Naïve Use

The STs who participated in this study often tried to use their students' mathematical thinking. This use, however, was not always effective. In analyzing their attempts to use students' mathematical thinking, we found a number of times when it was clear that the STs believed that they were indeed using the students' thinking effectively, although from our observation this was not the case. We classified such usage as naïve use—the STs were "technically" using their students' thinking, but such use was based on a naïve assumption about how students learn and did not really capitalize on the mathematical thinking of the students. The following sections describe the various types of naïve use that emerged from our analysis of the data.

Student Thinking as a Trigger. Using students' mathematical thinking as a trigger is somewhat akin to funneling. With funneling, however, the STs basically pass by all student thinking until they hear the thinking they are looking for—they then pursue or validate that thinking but fail to recognize the value in the other mathematical thinking that was shared. In the case of a trigger, the STs did recognize some value in what students' said, but the value is that they see a way that they can take some portion of what the students said (often not necessarily related to what the student meant) in order to redirect the conversation toward where they intended it to go. In terms of Woods' (1998) funneling and focusing constructs, when STs use students' mathematical thinking as a trigger they funnel but they *think* they are focusing—they think they are effectively using their students' thinking.

An example of using student thinking as a trigger comes again from the motion detector task and revolves around the issue of whether it was okay in this situation for the output numbers (distances) to be negative. In ST Megan's class she invited several students to put their answers on the board. Marcus then explained how they found the values in their table:

Marcus: He went to six, one, and then—it would usually be negative four, but it didn't say it was in front or behind. So we just thought it was four feet away from the [motion detector] because it went back up again.

ST Megan: Okay. Was that confusing to anyone else, whether or not you could go into the negatives? I saw a couple of papers where there was some argument. Let's talk about this idea of whether or not you can go into the negatives.

It appeared that ST Megan has listened to the student thinking that had been presented and that she recognized this thinking as worth pursuing. Rather than pursuing the reasoning of this pair of students, however, ST Megan chose to use their explanation as a trigger to talk about why it *does* make sense to use negative numbers in this situation. The students' explanation contained their reasoning for using positive numbers, not negative numbers. In fact, the mathematics of their explanation is focused on arguing that the context of the problem calls for the use of positive numbers. ST Megan viewed this as an incorrect answer and chose to use the statement "it would usually be negative four, but it didn't say it was in front or behind" as a trigger to first focus on the difficulty in deciding and then to focus on putting forth arguments that the values should be negative. Although the student's explanation was technically used in this situation, the mathematics of that explanation was neither used nor valued. Instead, a phrase about the problematic nature of the decision was taken up and used in order to redirect the focus of the discussion. We categorize such use of students' mathematical thinking as a trigger as naïve use. ST Megan seems to believe that she is indeed using the students' thinking, but her use is for her own purpose, which in this case is actually to try to make a point that is opposed the point the student was trying to make.

Mere presence of the correct solution. It was fairly common for the STs to elicit students' mathematical thinking and then fail to do anything with that thinking. In some cases, we concluded that the student thinking was merely elicited, but never listened to, let alone recognized as valuable and then used. In other instances, however, analysis revealed a variation on this phenomenon that we felt clearly should be classified as using (although naively) student thinking: the STs clearly believed that they were using the student thinking that had been elicited, although this use was at best implicit.

An episode from ST Megan's classroom illustrates this naïve use of student thinking. ST Megan had engaged her students in the Bicycle tour task (see Figure 1). For this part of the task, the students were asked to respond to the question, "Based on your graph, what price do you think the tour operators should charge? Explain" (Lappan et al., 2006, p. 32). After the students had worked on the task for some time, ST Megan initiated a class discussion. She asked the students to share their answers for part (c)—how much did they think should be charged and why. One pair of students said that they should charge $150 because it was at that price that the most customers had indicated that they would participate. ST Megan then asked for others to share their solution and a pair of students volunteered $350 (the correct solution) and explained that they used a revenue table to come up with their solution. A brief discussion followed about this latter answer, in which ST Megan implied that this latter answer was correct, and then the class moved on.

In this episode ST Megan elicited two different solutions with different solution methods and justifications. The first solution was incorrect; the second was correct. We classify this situation as a teachable moment because there are two reasonable solutions on the board and the class is primed to make arguments about their validity. Such a conversation would bring up the important mathematical ideas of revenue and maximization. ST Megan listened to the students' solutions and ensured that she understood them. Now, it is tempting to characterize ST Megan as not having recognized the teachable moment, and in terms of the explicit discussion and comparison of the two solutions, that might be correct. However, we believe that ST Megan actually thought that she was using both students' thinking. She facilitated the presentation of both solutions and she implicitly endorsed the latter (correct) response. We believe that ST Megan was operating under the following naïve assumption regarding student learning: the presence of the correct solution clears up the misconceptions underlying the incorrect solutions. It is by looking at ST Megan's teaching here through the lens of this assumption that we classify it as naïve use. Such naïve use of students' mathematical thinking seems to be based in a lack of KCS. This lack of knowledge allowed ST Megan to use here students' mathematical thinking, but only naively, thus hindering her from effectively (in this case, explicitly) using that thinking.

Mere presentation of multiple solutions. We briefly highlight a different but related naïve use of students' mathematical thinking that occurred fairly often in the STs' lessons. Sometimes the STs managed to elicit significant student thinking, had students record at the board and explain this thinking, and then the STs moved on. The goal of the lesson seemed to devolve into "student sharing," rather than developing mathematics from what the students were sharing, a phenomenon that has been previously noted in the literature (e.g., Ball, 2001; Silver, Ghousseini, Gosen, Charalambous, & Font Strawhun, 2005). Again, like before when the presence of the correct solution was interpreted as having cleared up the misconceptions underlying incorrect solutions, in these situations the STs seemed to be operating under the assumption that the connections between the multiple presented solutions, and the mathematics that could be derived from exploring those connections were evident to the students—that the students had learned important mathematics simply from being exposed to multiple solutions or solution strategies. Again it is under this assumption that we classify such an approach as naïve use of student thinking.

Incomplete Use

As we have mentioned previously, the STs often did manage to use the student thinking that they listened to and recognized as valuable. The extent to which they were able to use it effectively, however, even when they tried, was often limited by their limited knowledge. Consider the following episode:

The students in ST Megan's class were considering some data that reported the amount of time certain students spent watching TV and those students' GPA. ST Megan asked the class whether *TV Time* or *GPA* would be the independent variable. One student answered *TV Time* and ST Megan follows up:

ST Megan: What helped you decide it was *TV Time*?

Jamie: Because it was related to time.

ST Megan: Because it's related to time. [She notices another student with their hand raised.] Yes?

Nate: Time usually ends up as the independent variable and so it should go on the *x*-axis.

ST Megan recognized this student thinking as worth pursuing, as she had noticed that her students appeared to be choosing *time* as the independent variable almost automatically if it showed up as one of the variables. This recognition is evidenced by what ST Megan did next:

ST Megan: In some cases could it end up being the dependent variable? Do we have to be careful with that sometimes? What makes it hard to tell in this circumstance if its going to be independent or dependent? [pointing to a student who looks eager to contribute] Did you have an idea?

Jenna: Well, because there could be different situations where they could both be either independent or dependent.

ST Megan: Exactly. So, what kind of situation would we be thinking of if we said that *time* was dependent?

Taken together, Nate and Jenna's comments provided excellent student thinking on which ST Megan built toward a nice question that pushed her students to think more deeply about the mathematics of this situation. In particular, the class was poised for a discussion on determining the dependent and independent variables based on context. In the end, however, this conversation did not lead anywhere. The students were not able to think of a situation where *time* could be considered the dependent variable; most critically, neither was ST Megan able to construct such a situation. Because ST Megan lacked the SCK that would allow her to create such examples, she was not able to use effectively the student thinking in order to move the mathematical conversation towards her big mathematical idea—that the independent and dependent variables depend on the context (not just on whether *time* is one of the variables). ST Megan definitely recognized the shared student thinking as an opportunity to pursue important mathematical ideas, and she made a noble effort to do so. Her lack of SCK, however, hindered her ability to do so effectively and resulted in incomplete use of that thinking.

DISCUSSION AND CONCLUSION

Our analysis of the data revealed numerous roadblocks to the steps in the process of using students' mathematical thinking. By and large, these roadblocks can be characterized by a lack of teacher knowledge; the STs often lacked the SCK, PK, KCS or knowledge of curriculum to capitalize on teachable moments. At times, the STs recognized the value of their student's mathematical thinking and either did not pursue it or struggled to use it productively when they did pursue it because they did not have adequate knowledge of the representations or connections that would allow them to do so. At other times the STs heard and recognized student mathematical thinking that could be used to help all students better understand the topic at hand but did not have the PK to have a productive discussion about that thinking (like in the case of how to use three incorrect solutions in order to get at important mathematics).

Lack of curricular knowledge also often inhibited the STs from using student thinking. We hypothesize that the further removed a mathematical concept is from the lesson at hand, the less likely it is that STs will have the curricular knowledge to capitalize on a teachable moment that gets at that concept. The curriculum can be viewed as a set of concentric circles (see Figure 2), where the topic of the day is in the center. The goals of the unit, the course and mathematics in general are related to and often underlie the day's goal but often feel far removed to STs. STs' limited view of the curriculum (i.e., knowledge of curriculum) thus inhibits them from pursuing worthwhile mathematics.

In many of the roadblocks to effective use we have described in this paper, the STs recognized the teachable moment but lacked the knowledge (SCK, pedagogical, curricular) prevented them from using student thinking or only allowed minimal use. We identified a number of roadblocks, however, to the recognition of teachable moments. It seems as though KCS, in particular a knowledge of how students think about and learn mathematics, was the primary type of knowledge that inhibited this recognition. A common roadblock to recognition was an assumption of understanding, which comes from a lack of KCS. The STs' knowledge of students and how they learn led them to believe that once a concept had been "covered" (either by them or in a previous class) the students knew and understood that concept. If students seemed shaky with that knowledge, the STs tended to assume that the students had "just forgotten" what they had learned.

Another place where KCS had an influence on the productive use of student thinking was the naïve use. When the STs used the mere presentation of a correct solution as a way to clarify the flawed thinking that led to an incorrect solution, they exhibited their limited KCS. In this case, their KCS led them to believe that a student who had incorrectly solved a problem could resolve their misunderstanding by simply observing a correct solution, without having an explicit conversation about the approach. Similarly, the STs' KCS drove their approach of merely having the students share multiple solutions without any discussions of the connections between those solutions. Because of their lack of knowledge about how students learn, the STs assumed that the students would be able to see the similarities and differences between the different solutions without an explicit conversation about them.

It is interesting, however, to contrast this confidence in students' previous learning or their ability to make connections with the surprise (and doubt) that STs often express at the kinds of problems their students are able to solve. Thus STs tend to have high confidence in their students' previous learning abilities, but low confidence in their current learning abilities. Deeper KCS would likely foster quite the opposite set of assumptions about student learning, namely, a confidence in students' abilities to learn mathematics through solving problems on their own, but a healthy skepticism of their current mathematical understanding. Such skeptical optimism fosters an approach to teach wherein the teacher is very inquisitive about their students' thinking, always seeking to push on that thinking in the belief that such pushing will lead to great strides in student understanding.

The results of this study have important implications for mathematics teacher educators. First, our data demonstrate that STs are capable of learning to teach through focusing on their students' mathematical thinking. The revised student teaching structure in which these STs participated supported and encouraged their efforts to both elicit and use their students' thinking. These results add to a growing body of literature (e.g., Feiman-Nemser, 2001; Sowder, 2007) that refutes the logical fallacy that because novice teachers tend to begin with somewhat self-centered concerns, that teacher education programs should explicitly focus on addressing those concerns (Fuller, 1969). Our research demonstrates that STs are capable of focusing on and learning from their students' mathematical thinking.

Second, STs could benefit a great deal from an understanding of the steps of the process of effectively using student mathematical thinking: 1) listen and understand student thinking, 2) recognize the thinking as a teachable moment, and 3) use the thinking for a mathematical and pedagogical purpose. Although this process is certainly not the only way to teach in a way that is responsive to students' needs and thinking, the process does represent a tangible learning objective for novice teachers. Once the overall process is understood, STs are better able to reflect on their own teaching and that of others. They can focus on the points at which the process of using student thinking breaks down and on the type of knowledge that might help them to move further along in the process in the future. Also, as novice teachers better understand and value the process of using their students' mathematical thinking, they will realize that they need to plan into their lessons the time needed to pursue that thinking. It is difficult to discuss the mathematics of teachable moments if the time has not been allotted to do so.

Finally, teacher educators need to evaluate the degree to which their teacher education programs are designed to help novice teachers gain the knowledge needed to overcome these roadblocks. The content and structure of mathematics teacher education programs either afford or constrain the construction of this knowledge. Learning to teach activities, including open discussions with novice teachers about the pitfalls of an assumption of understanding could help them to begin to develop the skeptical optimism needed to recognize teachable moments. These discussions with novice teachers about how students learn mathematics could also help them to see the importance of making connections between solution strategies explicit through class discussions. Further activities designed to strengthen mathematical knowledge for teaching are then needed for novice teachers to develop the knowledge necessary to use those moments effectively.

Although much has been said about the importance of using significant mathematical tasks in order to elicit students' mathematical thinking, relatively little is known about the factors involved in using that mathematical thinking effectively. These results illustrate the complexity of this issue. More work needs to be done on designing and researching the effectiveness of "learning to teach" activities that can help novice teachers learn how to listen to, understand and recognize the value of their students' thinking, and then be able to use that thinking in order to orchestrate meaningful mathematical discussions. A good first step in this direction would be to discuss the process conceptualization and roadblocks presented here with preservice teachers as part of their teacher preparation program.

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$350	59
$400	49
$450	38
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$600	0

Figure 1. Table from Connected Mathematics 2 (Lappan et al., 2006, p. 32)

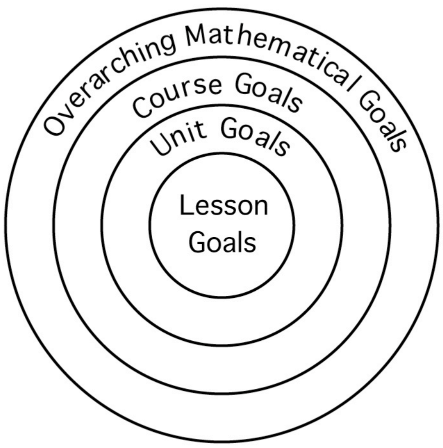

Figure 2. A representation of the broadening layers of the mathematics curriculum.