2 *Sum-Thing To Talk About*

Important Stuff

As you learned yesterday, the sum of the divisors of a number is cool. So, the sum of the divisors' *reciprocals* has to be ice cold! Today Todd, Todd, and the rest of our investigatory team tackle a function called a, which takes in a positive integer, and spews forth the sum of its divisors' reciprocals. For example, $a(12) = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{12} = \frac{7}{3}$.

This is known as André 3000's Theorem, but he's a bit of a mathematical outkast.

PROBLEM

Here's a table for the a function. Complete the table without using anything that can calculate faster than you. Write answers in "lowest terms."

n	1	2	3	4	5	6	7	8	9	10	11	12
$a(n)$												$\frac{7}{3}$

n	13	14	15	16	17	18	19	20	21	22	23	24
$a(n)$												

n	25	26	27	28	29	30	31	32	33	34	35	36
$a(n)$												

n	37	38	39	40	41	42	43	44	45	46	47	48
$a(n)$												

See, you still only have to write 96 numbers, since there are 48 numerators and 48 denominators. Aren't we nice? Hey, it wasn't even 96 numbers!

pssst... there's more fun to be had on the next page!

Watch this space for important messages. Eventually there might be one. . .

1. Determine each of the following.

(a) $a(3) \cdot a(4)$

(b) $a(2) \cdot a(5)$

(c) $a(8) \cdot a(15)$

(d) $a(120)$

(e) $a(10) \cdot a(12)$

Um, shouldn't this be problem #2? Yeah yeah, sure sure.

2. Determine each of the following. No calculator please!

(a) $\left(1+\frac{1}{3}\right)\left(1+\frac{1}{2}+\frac{1}{4}\right)$

(b) $\left(1+\frac{1}{2}\right)\left(1+\frac{1}{5}\right)$

(c) $\left(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}\right)\left(1+\frac{1}{3}+\frac{1}{5}+\frac{1}{15}\right)$

(d) $\left(1+\frac{1}{2}+\frac{1}{5}+\frac{1}{10}\right)\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{6}+\frac{1}{12}\right)$

3. Calculate each of the following.

(a) $1+\frac{1}{2}$

(b) $1+\frac{1}{2}+\frac{1}{4}$

(c) $\frac{1}{2^0}+\frac{1}{2^1}+\frac{1}{2^2}+\frac{1}{2^3}$

(d) $\frac{1}{2^0}+\frac{1}{2^1}+\frac{1}{2^2}+\cdots\frac{1}{2^6}$

(e) The sum of all numbers in the form $\frac{1}{2^n}$ as n goes from 0 to 10

(f) $\displaystyle\sum_{n=0}^{11}\frac{1}{2^n}$

(g) $\displaystyle\sum_{n=0}^{\infty}\frac{1}{2^n}$

Zack points out that because it says "calculate", you can use a calculator now!

Sigma is your friend. Or at least, it wants to be your friend. Σ is σ's daddy.

4. Let n be a power of 3. Find the *smallest* possible number k for which you are completely sure that $k > a(n)$.

Neat Stuff

5. For certain values of n, it turns out that $\sigma(n) = 3 + \frac{n}{2} + n$. Classify these numbers and find a generalization.

Remember, the σ function is the sum of the divisors. Or is it? No, it is.

6. If p and q are primes, write a rule for $\sigma(pq)$ in terms of p and q.

7. If p and q are primes, write a rule for $a(pq)$ in terms of p and q and give the simplest answer you can.

8. Let n be a number whose factors consist only of 2s and 3s.

(a) Find the smallest possible number k for which you are completely sure that $k > a(n)$.

(b) Find a suitable number n such that $k - a(n) < 0.1$.

9. Hillary challenges you to find a number for which $\sigma(n) = 1000$ or show that no such number exists.

Please don't disappoint Hillary by using technology to answer this question.

10. Find the maximum possible value of $a(n)$.

11. Find the first ten numerators in the following bizarre-looking expansion. Do *not* try to simplify or combine terms, just expand!

$$\left(\frac{1}{1^x} + \frac{2}{2^x} + \frac{3}{3^x} + \frac{4}{4^x} + \cdots\right)\left(\frac{1}{1^x} + \frac{1}{2^x} + \frac{1}{3^x} + \frac{1}{4^x} + \cdots\right)$$
$$= \frac{?}{1^x} + \frac{?}{2^x} + \frac{?}{3^x} + \frac{?}{4^x} + \cdots$$

12. Define $\sigma_2(n)$ to be the sum of the squares of the divisors of n and $b(n)$ as the sum of the reciprocals of the squares of the divisors of n.

(a) Tabulate the σ_2 function from 1 to 10.

(b) Find some interesting things about the σ_2 function.

(c) Calculate $\sigma_2(120)$ as easily as possible. Without a calculator please.

(d) Find a number with $b(n) = 2$, or prove that no such number exists.

That's way too many "of the"s for a reasonable sentence. Sure would be nice to have some notation to simplify these kinds of descriptions. Of the.

13. Find the first ten numerators in the following bazaar-looking expansion. Do *not* try to simplify or combine terms, just expand!

$$\left(\frac{1}{1^x} + \frac{4}{2^x} + \frac{9}{3^x} + \frac{16}{4^x} + \cdots\right)\left(\frac{1}{1^x} + \frac{1}{2^x} + \frac{1}{3^x} + \frac{1}{4^x} + \cdots\right)$$
$$= \frac{?}{1^x} + \frac{?}{2^x} + \frac{?}{3^x} + \frac{?}{4^x} + \cdots$$

Tough Stuff

14. Find all numbers for which $\sigma(n) = 1000$.

15. Find a number n for which $a(n) \geq 5$, or prove that no such number exists.

16. Find an odd number for which $a(n) = 2$, or prove that no such number exists.

17. Find the maximum possible value of $b(n)$.

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