Jokes? *Jokes?* We don't need no stinkin' jokes!

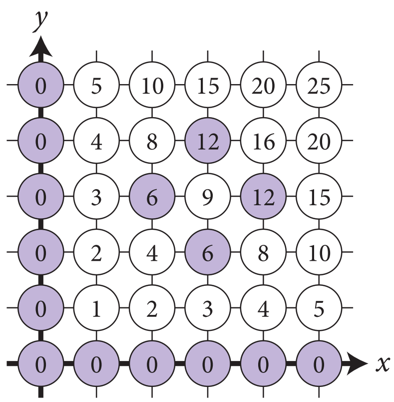
3 *Dim Sum*

Important Stuff

The figure below tabulates the product xy for different values of x and y. How many of these numbers are multiples of 6?

Today, Kim, Kym, and our i-team investigates $k(n)$, which takes in an integer and expectorates the number of products xy that are multiples of n for x and y ranging from 0 to $n-1$.

Err... Did you notice that the multiples of 6 are shaded in the figure below? Yep, 0 is a multiple of 6.



Based on the picture above, $k(6) = 15$.

PROBLEM

Here's a table for the k function. Complete the table without using anything containing silicon. Use the handout to help you.

n	1	2	3	4	5	6	7	8	9	10	11	12
$k(n)$	1					15						

n	13	14	15	16	17	18	19	20	21	22	23	24
$k(n)$												

1. Notice anything interesting about $k(n)$?

Sorry, but "Nope" is not an acceptable answer here.

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2. (a) Armando says that if a number is a multiple of 8 and a multiple of 15, then it must be a multiple of 120. What do you think?

(b) Jennifer says that if a number is a multiple of 10 and a multiple of 12, then it must be a multiple of 120. What do you think?

Which Jennifer? We're not telling.

3. Use the units digit of each number to determine which of the following multiplications was performed *incorrectly*.

(a) $234 \times 153 = 35802$

(b) $157 \times 321 = 50397$

(c) $223 \times 155 = 34565$

(d) $168 \times 183 = 30746$

4. When working with units digits you are working in "mod 10", a number system that considers only remainders when dividing by 10. In mod 10, the only numbers are 0 through 9. There are lots of other "mods" too.

If all these mods got together to form a team, it would be the Mod . . . Cabal? Something like that.

(a) What is 3×4 in mod 10?

(b) What is 8×3 in mod 10?

(c) What is 8×3 in mod 12?

(d) What number x solves $2x = 7$ in mod 9?

(e) What *two* numbers x solve $x^2 = 7$ in mod 9?

The answer isn't $\sqrt{7}$ here! The only numbers are 0 through 8. But the answers do "act like" $\sqrt{7}$ in some way.

5. Consider this summation:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \underbrace{\frac{1}{16} + \cdots + \frac{1}{16}}_{\text{8 of these}} + \cdots$$

where each term is repeated 1, then 2, then 4, then 8 times, and it goes on forever. What happens to the sum as you take more terms? Is there a limit to the maximum value of the sum?

This problem was featured in the Johnny Lott biopic "Walk the Number Line."

6. A function f is called *multiplicative* if $f(ab) = f(a) \cdot f(b)$ whenever a and b don't share any common factors higher than 1.

Felipe calls this situation "relatively prime," which is the more technical term for not sharing common factors. So if you hear that, it means the other thing.

(a) Give three examples of multiplicative functions you've seen in this course.

(b) Give three more examples of multiplicative functions.

Did you re-read the first page from Day 1 like we said you should? Good for you!

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Neat Stuff

7. Demonstrate each of the following, any way you like.

(a) $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots=2$

(b) $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots=\frac{3}{2}$

(c) $\frac{1}{4^0}+\frac{1}{4^1}+\frac{1}{4^2}+\frac{1}{4^3}+\cdots=\frac{4}{3}$

(d) $\sum_{n=0}^{\infty}\frac{1}{5^n}=\frac{5}{4}$

(e) $\sum_{n=0}^{\infty}\frac{1}{p^n}=\frac{p}{p-1}$

This last bit isn't actually true all the time! Feel free to investigate but we will only be working with numbers that make this true.

8. Remember $a(n)$ from yesterday? Sure you do! Consider $a(72)$.

(a) What is the value of $a(72)$?

(b) Calculate

$$\left(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}\right)\left(1+\frac{1}{3}+\frac{1}{9}\right)$$

(c) *Without performing the final addition*, expand the expression above. What happens?

One of the terms in the expansion is $\frac{1}{18}$.

9. The function $\tau(n)$ is defined as the number of factors of n.

(a) Tabulate the τ function for $n=1$ through 20.

(b) Is τ multiplicative? Can you prove it?

(c) Describe a way to calculate $\tau(n)$ for any integer n.

10. Let $n=2^p3^q$. Find some values of p and q that produce particularly large values of $a(n)$, then determine the maximum possible value of $a(n)$ for any number in this form.

11. Let $n=2^p3^q4^r$. Determine the maximum possible value of $a(n)$ for any number in this form.

12. Let $n=2^p3^q5^r$. Determine the maximum possible value of $a(n)$ for any number in this form.

13. Consider this summation:

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\cdots$$

What happens to the sum as you take more terms? Is there a limit to the maximum value of the sum?

14. Prove that for any n, $a(n!)$ is larger than the sum of the first n terms in the series of Problem 13.

You may find something helpful in a previous problem, and the "number line jumping" picture may also be useful.

15. Prove that there is no maximum value of $a(n)$.

16. Find the first ten numerators in each of these bizah-looking expansions. Or more than 10, we don't care. Notice anything?

For example, when you end up seeing a term like $\frac{1}{2^x 3^x}$, that's really $\frac{1}{6^x}$. But don't try to simplify something like $\frac{2}{2^x}$ to $\frac{1}{2^{x-1}}$, just leave it so the denominators are all k^x.

(a)

$$\left(\frac{1}{1^x}+\frac{1}{2^x}+\frac{1}{3^x}+\frac{1}{4^x}+\cdots\right)\left(\frac{1}{1^x}+\frac{1}{2^x}+\frac{1}{3^x}+\frac{1}{4^x}+\cdots\right)$$
$$=\frac{?}{1^x}+\frac{?}{2^x}+\frac{?}{3^x}+\frac{?}{4^x}+\cdots$$

(b)

$$\left(\frac{1}{1^x}+\frac{2}{2^x}+\frac{3}{3^x}+\frac{4}{4^x}+\cdots\right)\left(\frac{1}{1^x}+\frac{1}{2^x}+\frac{1}{3^x}+\frac{1}{4^x}+\cdots\right)$$
$$=\frac{?}{1^x}+\frac{?}{2^x}+\frac{?}{3^x}+\frac{?}{4^x}+\cdots$$

(c)

$$\left(\frac{1}{1^x}+\frac{1/2}{2^x}+\frac{1/3}{3^x}+\frac{1/4}{4^x}+\cdots\right)\left(\frac{1}{1^x}+\frac{1}{2^x}+\frac{1}{3^x}+\frac{1}{4^x}+\cdots\right)$$
$$=\frac{?}{1^x}+\frac{?}{2^x}+\frac{?}{3^x}+\frac{?}{4^x}+\cdots$$

17. Let $f(n)$ be the number of solutions to $x^2 = 7$ in mod n. Figure out anything you can about this function.

If $n < 7$ you will need to adjust the equation to suit the mod. For example, in mod 5, the equation becomes $x^2 = 2$.

18. Define $\sigma_2(n)$ to be the sum of the squares of the divisors of n and $b(n)$ as the sum of the reciprocals of the squares of the divisors of n.

(a) Tabulate the σ_2 function from 1 to 10.

(b) Find some interesting things about the σ_2 function.

(c) Find $\sigma_2(120)$ without a calculator.

Tough Stuff

19. Find a number n for which you can prove *without use of any technology* that $a(n) > 10$.

One such number was found yesterday, by brute force calculation—it was over 200 digits long. Probably the one you find here will be much longer.

20. For what primes p is 7 a perfect square in mod p?

21. Find the maximum possible value of $b(n)$, where b is the function from Problem 18.