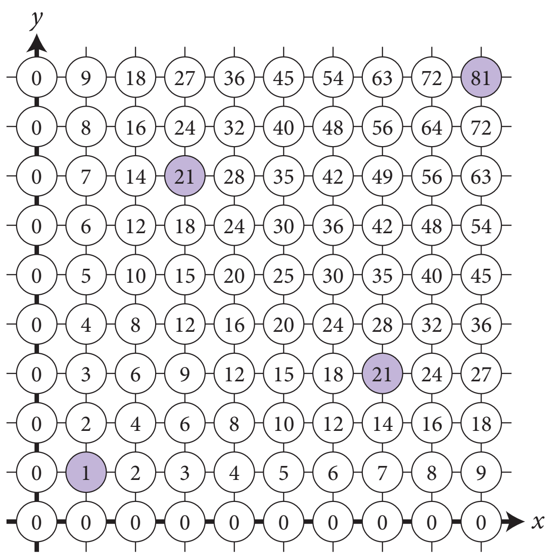
Congrats to Bill on his successful singing of James Brown's "Get On Up"!

4 *Sum-thing Wicked This Way Comes*

Important Stuff

The figure below tabulates the product xy for values of x and y from 0 to 9. How many of these products end in 1? All these numbers are 1 more than a multiple of 10. Today Emily and our mod squad investigate $P(n)$, which takes in an integer and regurgitates the number of products xy that are *one more* than a multiple of n. Like yesterday, x and y range from 0 to $n - 1$.

So, you can use the same table as yesterday, but we've provided lots of copies with that annoying diagonal filled in too!



Based on the picture above, $P(10) = 4$, since there are four numbers that are 1 more than a multiple of 10.

Remember, for $P(9)$ you'd want to find numbers that are 1 more than a multiple of 9.

PROBLEM

Here's a table for the P function. Complete the table without using anything that has an "enter" key. Use the handout to help you, and don't forget to cheat appropriately.

n	1	2	3	4	5	6	7	8	9	10	11	12
$P(n)$	1		2					4		4		

n	13	14	15	16	17	18	19	20	21	22	23	24
$P(n)$												

1. Notice anything interesting about $P(n)$?

It's okay to say "nah," but then you still gotta find sum-thing.

2. Describe how you could directly find $P(n)$ for any n, then use your method to find $P(210)$.

3. A function f is called *multiplicative* if $f(ab) = f(a) \cdot f(b)$ whenever a and b don't share any common factors higher than 1.

If a and b share common factors, all bets are off!

(a) Give *four* examples of multiplicative functions you've seen in this course.
(b) Give one more example of a multiplicative function.
(c) Give three examples of functions of functions that are *not* multiplicative.

4. (a) Katie says that if a number is one more than a multiple of 8 and one more than a multiple of 15, then it must be one more than a multiple of 120. What do you think?
(b) Cliff says that if a number is one more than a multiple of 10 and one more than a multiple of 12, then it must be one more than a multiple of 120. What do you think?

A déjà vu is usually a glitch in some matrices. It happens when Bowen and Darryl change something.

5. Functions can have babies! Define the *child* g of a function f by the following:

$g(\text{Aaron}) = f(\text{Bowen}) + f(\text{Nancy})$

$$\begin{aligned} g(n) &= f(\text{all divisors of } n) \text{ added together} \\ g(15) &= f(1) + f(3) + f(5) + f(15) \\ g(20) &= f(1) + f(2) + f(4) + f(5) + f(10) + f(20) \\ g(1) &= f(1) \end{aligned}$$

Let $r(n) = n$, and let s be the child of r.
(a) Calculate $s(1)$ through $s(10)$, $s(15)$, and $s(20)$.
(b) Is r multiplicative? Does s seem to be multiplicative?
(c) Hey, where have we seen s before?

6. Find all solutions to these equations. Reminder: in mod 7, the only possible answers are 0, 1, 2, 3, 4, 5, and 6.
(a) $3x = 4$ in mod 7
(b) $6x = 4$ in mod 7
(c) $6x = 4$ in mod 8
(d) $6x = 1$ in mod 8
(e) $x^2 = 2$ in mod 7
(f) $x^3 = 1$ in mod 7
(g) $x^3 = -1$ in mod 7
(h) $x^6 = 1$ in mod 7

Just like there is more than one John in the room, there can be more than one answer to each question. Or no answer at all.

Wait, -1 doesn't exist in mod 7...oh right, it's 6.

Neat Stuff

7. Many pairs of numbers have no common factor higher than 1: for example, 8 and 15. Function $\phi(n)$ returns the number of values from 1 to n that, when checked against n, have no common factor higher than 1.

(a) Show that $\phi(3) = 2$, $\phi(5) = 4$, and $\phi(15) = 8$.

(b) Calculate values of the ϕ function until you figure out what is happening.

How do *you* pronounce that greek letter ϕ? Is it "phee, phi, pho, phum" or "phi, phy, pho, phum"? Only your hairdresser knows for sure!

8. Let $S = 1 + 5 + 5^2 + 5^3 + \cdots + 5^n$.

(a) Write an expression for $5S$.

(b) Write a really clever expression for $4S$ by subtracting.

(c) Show that

$$S = \frac{5^{n+1} - 1}{4}$$

(d) Find a general rule for $1 + r + r^2 + r^3 + \cdots + r^n$.

9. What is the child of $f(n) = 1$?

10. Remember function k from yesterday? Find the child of k and tabulate its values. What patterns do you observe?

Sometimes children are easier to understand than parents. . .

11. Prove that the sum of the harmonic sequence

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{n} + \cdots$$

does not converge to any real number. Bonus points if you use a method different from the one presented today.

12. Remember function a? Write out the largest eight fractions that are part of each of these.

(a) $a(24)$

(b) $a(720)$

(c) $a(10!)$. . . make it the largest twelve fractions here.

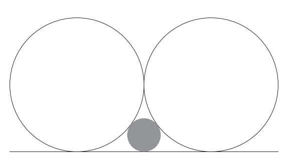
13. Prove that for any n, $a(n!)$ is larger than the sum of the first n terms in the series of Problem 11.

Pronounce this function as "a of $ENNN$!"

14. Prove that there is no maximum value of $a(n)$.

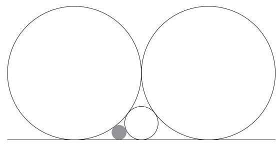
15. Two circles with diameter 1 are tangent to a line, as well as to each other. What is the diameter of the gray circle that is tangent to both circles as well as the line?

And now, for something completely different! *Or is it?* No, it is.



Have an awe-sum weekend!

16. Start with the diagram from the previous problem, and add one more circle in the hole in the lower left; it is also tangent to the line and two other circles. What is the diameter of this circle (marked in gray below)?



17. (a) Find an equation that has *every number* as a solution in mod 7.

(b) Multiply this out and simplify it in mod 7:

$$x(x-1)(x-2)(x-3)(x-4)(x-5)(x-6)$$

You might want to use a CAS calculator for this. Get a new calculator window using the HOME icon in the upper right, then select "expand" from the menus. "Simplify in mod 7" means that if you see 9x, write 2x.

18. Let $f(n)$ be the number of solutions to $xy = 5$ in mod n. Figure out anything you can about this function.

19. Hey, go work on that sum of the squares of divisors problem from earlier.

Tough Stuff

20. Prove that the P and ϕ functions must be identical. Specifically, prove that if a is relatively prime to n, then there is exactly one solution to $ab = 1 \bmod n$, and if a isn't relatively prime, there are no solutions.

21. For what primes p is 5 a perfect square in mod p?

22. What is the *grandparent* of $f(n) = 1$?

Eww, we didn't tell you how to find a parent if you're given the child... and you have to do that twice!

23. $b(n)$ is the sum of the squares of the reciprocals of the divisors of n. Find the maximum possible value of $b(n)$ or prove there is no maximum.

24. Two circles with diameters a and b are tangent to a line, as well as to each other. How does the diameter c of the gray circle compare? Come up with an incredibly nice relationship between a, b, and c.

Is it symmetric in a, b, c? Better be or it's not nice enough!