6 *Children of M(n)*

Hey, welcome to the class. We know you'll learn a lot of mathematics here—maybe some new tricks, maybe some new perspectives on things with which you're already familiar. A few things you should know about how the class is organized:

This is reprinted from Day 1, since we have many new people in the room to start Week 2. Hope you had a great weekend! Also, we hope you're a fan of Cuarón or the title probably makes no sense.

- **Don't worry about answering all the questions.** If you're answering every question, we haven't written the problem sets correctly.
- **Don't worry about getting to a certain problem number.** Some participants have been known to spend the entire session working on one problem (and perhaps a few of its extensions or consequences).
- **Stop and smell the roses.** Getting the correct answer to a question is not a be-all and end-all in this course. How does the question relate to others you've encountered? How did others at your table think about this question?
- **Respect everyone's views.** Remember that you have something to learn from everyone else. Remember that everyone works at a different pace.
- **Learn from others.** Give everyone the chance to discover, and look to your tablemates for new perspectives on problems. Resist the temptation to tell others the answers if they aren't ready to hear them yet. If you think it's a good time to teach your tablemates about dividing by the zeta function, think again: the problems should lead to the appropriate mathematics rather than requiring it. The same goes for technology: the problems should lead to the appropriate use of technology rather than requiring it.
- **Each day has its Stuff.** There are problem categories: Important Stuff, Neat Stuff, Tough Stuff, and maybe other stuff sometimes. Check out Important Stuff first. All the mathematics that is central to the course can be found and developed in the Important Stuff. *That's* why it's Important Stuff. Everything else is just neat or tough. If you didn't get through the Important Stuff, we noticed... and the important questions recycle. Each problem set is based on what happened before, either in problems or in class discussions.

We promise to add more unsolved problems to the two that have already appeared on the problem sets.

Resist! Resist, we say!

Important Stuff

PROBLEM

Last week we learned how functions make babies. The general rule is that

child(n) = parent(all divisors of n) added together

Today's amazing parent function is $m(n) = 1$. Yup, you got it. It's a function that always gives you the number 1. It's not a particularly interesting function but it has a fabulous family!

Let t be the child of m and let u be the child of t. This means that

$$\begin{aligned} t(n) &= m(\text{all divisors of } n) \text{ added together} \\ u(n) &= t(\text{all divisors of } n) \text{ added together} \\ t(1) &= m(1) = 1 \\ t(15) &= m(1) + m(3) + m(5) + m(15) = 4 \\ u(16) &= t(1) + t(2) + t(4) + t(8) + t(16) = ?? \end{aligned}$$

Fill in this table with the values of $t(n)$ and $u(n)$.

n	1	2	3	4	5	6	7	8	9	10	11	12
$m(n)$	1											
$t(n)$	1											
$u(n)$	1											

n	13	14	15	16	17	18	19	20	21	22	23	24
$m(n)$												
$t(n)$			4									
$u(n)$												

Andy said it had something to do with a stork and an input-output table...

What is $m(7)$? What is $m(m(7))$? What is $m(Ana)$? You get the idea.

1. Is t multiplicative? How about u? m?

2. You may have noticed in the above problem that primes play a large role in determining the values of these functions. Complete this shorter table for each function.

n	$m(n)$	$t(n)$	$u(n)$
1			
p			
p^2			
p^3			
p^4			

Um?... how about *you*? A function is *multiplicative* if $f(ab) = f(a) \cdot f(b)$ whenever a and b don't have a common factor greater than 1.

3. Use the tables from Problem 2 to quickly compute $m(420)$, $t(420)$, and $u(420)$.

420, eh? Hopefully this problem won't leave you dazed and confused. Sorry, that was a half-baked reference.

4. (a) If n is one more than a multiple of 3 and one more than a multiple of 8, what's the most you can say about the value of n?

(b) If n is 2 more than a multiple of 3 and 7 more than a multiple of 8, what's the most you can say about the value of n?

5. On Friday, Assegid mentioned that many perfect squares appeared to be one more than a multiple of 24. For what values of n, from $1 \le n \le 48$, is n^2 one more than a multiple of 24?

A calculator might be handy; those squares get big. Or use the "double plus 1" rule:

$$15^2 = 14^2 + (2 \cdot 14 + 1)$$

$$16^2 = 15^2 + (2 \cdot 15 + 1)$$

6. Find all solutions to each of the following.

(a) $x^2 = 1$ in mod 3
(b) $x^2 = 1$ in mod 8
(c) $x^2 = 1$ in mod 24
(d) $x^2 = 1$ in mod 72

7. Last week you worked with the ϕ function, which counts how many numbers between 1 and n have no common factors with n (larger than 1). For example, $\phi(15) = 8$ because the numbers 1, 2, 4, 7, 8, 11, 13, and 14 all have no common factors with 15. (Note that $\phi(1) = 1$, not 0.)

(a) List all the numbers that make up $\phi(1), \phi(3), \phi(7)$, and $\phi(21)$.

(b) What is $\phi(1) + \phi(3) + \phi(7) + \phi(21)$?

(c) Simplify this expression as much as possible:

$$1 + (p - 1) + (q - 1) + (p - 1)(q - 1)$$

And by "you" we mean *you*, Lalit. Okay, maybe we mean everyone who was here last week!

Neat Stuff

8. Complete this table for some "fan favorite" functions from Week 1. $\sigma(n)$ is the sum of the factors of n, $a(n) = \frac{\sigma(n)}{n}$, and $\phi(n)$ is given in Problem 7.

n	$\sigma(n)$	$a(n)$	$\phi(n)$
1			
p			
p^2			
p^3			
p^4			

Answers here are in terms of p, so you might consider testing some different choices of p first. But not $p = 4 \ldots$

9. If v is the child of u, what would its column in the table from Problem 2 look like? Compute $v(420)$.

Another 420?! Hopefully this time it's just a reference to blackbirds baked in a pie.

10. (a) If x is 1 mod 3, 0 mod 5, and 0 mod 7, what is it in mod 105?
(b) If y is 0 mod 3, 1 mod 5, and 0 mod 7, what is it in mod 105?
(c) If z is 0 mod 3, 0 mod 5, and 1 mod 7, what is it in mod 105?
(d) Compute $2x + 3y + 4z$ in mod 105.

11. If x is 2 mod 3, 3 mod 5, and 4 mod 7, what is it in mod 105?

12. Find all solutions to each of the following. Don't use a calculator unless you feel really, really compelled to.

(a) $x^2 = 1$ in mod 3
(b) $x^2 = 1$ in mod 5
(c) $x^2 = 1$ in mod 7
(d) $x^2 = 1$ in mod 15
(e) $x^2 = 1$ in mod 21
(f) $x^2 = 1$ in mod 35
(g) $x^2 = 1$ in mod 105
(h) $x^2 = 1$ in mod 420

13. Find all solutions to each of the following. See previous rant about calculator usage.

Vicki claims the result in part (a) is very helpful in some later parts. And she's right!

(a) $x^2 = 2$ in mod 3
(b) $x^2 = 2$ in mod 7
(c) $x^2 = 2$ in mod 17
(d) $x^2 = 2$ in mod 21
(e) $x^2 = 2$ in mod 51
(f) $x^2 = 2$ in mod 119
(g) $x^2 = 2$ in mod 357... which is $3 \times 7 \times 17$

14. Prove that if f is multiplicative, then either $f(1) = 1$ or $f(n) = 0$ all the time.

15. Prove that if f and g are multiplicative functions, then $h = fg$, the product, is also multiplicative.

16. Find a value of n for which $x^2 = 2$ has exactly 8 solutions in mod n.

It's over there! Scavenger hunt bonus: actually *find* this number.

17. Define $c(x) = \frac{\phi(x)}{x}$.
(a) Explain why $c(x) < 1$ as long as $x > 1$.
(b) Find a value for which $c(x) < 0.1$ or show there is no such x.
(c) Find the minimum possible value of $c(x)$.

18. We said $m(n) = 1$ had a fabulous family, but only showed the children!
(a) Find the parent of m; that is, a function d so that m is the child of d.
(b) Find z, the parent of d.

Think of the children!! Won't *someone* think of the children?!

19. Prove that if f is multiplicative, then so is $\frac{1}{f}$, the reciprocal, as long as $f(n) \neq 0$ for any n.

20. Prove that the child of ϕ is the identity $r(n) = n$.

21. Find a rule or give a formula for each function.
(a) $f(m)$ is the number of solutions to $x^2 = x$ in mod m.
(b) $g_1(m)$ is the number of solutions to $xy = 1$ in mod m.
(c) $g_2(m)$ is the number of solutions to $x^2 - y^2 = 1$ in mod m.
(d) $h_1(m)$ is the number of solutions to $x^2 - y^2 = 0$ in mod m.
(e) $h_2(m)$ is the number of solutions to $x^2 + y^2 = 0$ in mod m.

Tough Stuff

22. Under what conditions is 2 a perfect square in mod m? Be careful about composite m...

Go away! Stop working on the Tough Stuff, especially after hours!

23. Prove that if g is the child of f, and f is multiplicative, then g is multiplicative.

24. Prove that if g is the child of f, and f is *not* multiplicative, then g is *not* multiplicative.

25. Can a nonzero multiplicative function be its own ancestor (allowing for more than one generation)?

26. Let $s_4(n)$ be the number of ways to write n as the sum of four squares: $n = a^2 + b^2 + c^2 + d^2$ where a, b, c, d are integers (perhaps zero or negative). Starting with $s_4(1) = 8$, tabulate s_4 from 1 to 24. Is s_4 multiplicative? Fix it!

27. This identity shows that if m and n can be written as the sum of two squares, then mn can be written as the sum of two squares:

$$(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2$$

Find an identity that shows the same fact when m and n are each written as the sum of four squares.

Clearly, a problem like this requires the use of quaternions. What are those? *I* don't know, but ask MIB agents *J* and *K*.

Alternate title for today: One Function Per Child

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