Best wishes to those suffering from Sine Flu.

8 *Problem Children*

Important Stuff

An important Calendar Event will happen during class today! Do you know what it is?

PROBLEM

We're off to solve the equation $x^3 - 1 = 0$ in mod 63! Here are the steps:

- Find all solutions to $x^3 - 1 = 0$ in mod 7.
- Find all solutions to $x^3 - 1 = 0$ in mod 9.
- Use a grid technique to find all solutions in mod 63.

So, off you go; make a big O around any solution to $x^3 - 1 = 0$ in mod 7 (and its equivalent numbers), write an X over any solution to $x^3 - 1 = 0$ in mod 9 (and its equivalent numbers).

0	1	2	3	4	5	6	7	8
9	10	11	12	13	14	15	16	17
18	19	20	21	22	23	24	25	26
27	28	29	30	31	32	33	34	35
36	37	38	39	40	41	42	43	44
45	46	47	48	49	50	51	52	53
54	55	56	57	58	59	60	61	62

Let $f(n)$ be the number of solutions to your equation in mod n, remembering that the actual numbers in mod n go from 0 to $n-1$. What is $f(7)$? What is $f(9)$? What is $f(63)$?

We're off to solve the 'quation, the wonderful 'quation of mods...

XOXO Gossip Girl! What, come on, you don't watch Gossip Girl? Anyway, if 4 is an answer to $x^3 - 1 = 0$ in mod 7, you'd circle 4, 11, 18, 25... A number is a multiple of 9 whenever its digits add up to a multiple of 9.

Make sure $f(7)$ isn't larger than 7... that would be bad. Total protonic reversal bad.

1. (a) If a number is a multiple of 63, is it *required* to be a multiple of 7 and a multiple of 9?

(b) If a number is a multiple of 7 and a multiple of 9, is it *required* to be a multiple of 63?

2. (a) If a number is a multiple of 60, is it *required* to be a multiple of 6 and a multiple of 10?

(b) If a number is a multiple of 6 and a multiple of 10, is it *required* to be a multiple of 60?

3. If you didn't get the chance to do yesterday's circle problem, try it now. Use the grid (counting boxes) to find the diameters and centers of the two smaller circles.

Each box is $\frac{1}{36}$ on a side. Why would we pick such a strange scale??

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4. A wicked awesome formula for dealing with yesterday's circle problem was discovered by Descartes, forgotten, then rediscovered in Japan in the early 19th Century.

Suppose three circles are all mutually tangent, and tangent to the same line. If the *reciprocals* of the circles' diameters are a, b, c then, astoundingly:

$$2(a^2 + b^2 + c^2) = (a + b + c)^2$$

(a) Let $a = 1$ and $b = 1$. Use the equation above to find both possible values of c.

(b) Use c to determine the diameter of the first small circle.

(c) Let $a = 1$ and $b = 4$. Use the equation to find both possible values of c.

(d) Find the diameter of the second small circle.

(e) Find the diameter of the third small circle. (What should a and b equal?)

(f) Find the diameter of the fourth small circle, and the fifth.

Apparently, Descartes was from Revere, Massachusetts. Who knew?

Ooh, symmetric. Don't forget: these are the *reciprocals*. A formula using the actual diameters can be written, but it's not as wickedly awesome.

Wait, in part (b) there are two values of c. What does the other value of c give for the diameter? What would a circle with that diameter look like??

5. Look back at yesterday's set and work through Problems 5 and 6 if you haven't already.

6. A rock-paper-scissors player makes their selection randomly. In five games, they picked paper once, rock three times, and scissors once. In how many different ways could this have happened?

Good ole rock, nothin' beats rock. One way is PRRRS, another is RPRSR.

7. Using an Nspire or other algebraic calculator, compute the expansion of $(p + r + s)^5$ to answer the previous question.

Returning participants may prefer to expand $(b + n + c)^5$ for an alternate version of the game. Ask Bill or Darryl.

Neat Stuff

8. (a) If x is 1 in mod 7 and 0 in mod 9, what is it in mod 63?

(b) If y is 0 in mod 7 and 1 in mod 9, what is it in mod 63?

(c) Compute the following nine quantities, giving answers in mod 63:

$$\begin{array}{ccc} x+y & 2x+y & 4x+y \\ x+4y & 2x+4y & 4x+4y \\ x+7y & 2x+7y & 4x+7y \end{array}$$

What do you notice?

What's In The Box?!

9. Describe a method to solve *any* equation in mod 63.

10. Describe a method to predict the number of solutions to any equation in mod 63.

11. Define $c(n) = \frac{\phi(n)}{n}$.
(a) Make a table for c from 1 to 36, using exact fractional answers.
(b) Complete this table for $c(n)$ looking at powers of primes.

Pick a number for p if you're having trouble. *No, not that number!!!* Ahh, just kidding.

n	$c(n)$
1	
p	
p^2	
p^3	
p^4	

(c) Do you think it is possible for $c(n)$ to be less than 0.1? Explain.

12. Let $d(n) = a(n) \cdot c(n)$, where $a(n)$ is the sum of the reciprocals of the factors of n (see Day 2) and $c(n)$ is defined in Problem 11.
(a) Compute $d(n)$ using any method from 1 to 36, giving decimal answers to four places.
(b) Does d seem to have a maximum value? A minimum?

We named this function after Dianna. Or maybe Dawn. Or Darryl?

13. Solve $x^2 + x = 6$ in mod 105.

14. How many solutions are there to $x^2 = 1$ in mod 1155?

15. Adam says that you can identify a multiplicative function just by declaring what it does to powers of primes. Nobody believes him, but it's true! For each description, give a simple rule for the function f.
(a) $f(p^k) = k + 1$
(b) $f(p^k) = 1$
(c) $f(p^k) = p^k$
(d) $f(p^k) = 1 + p + p^2 + \cdots + p^k$

16. Complete this table for $\phi(n)$ and its child.

Remember, the *child* of a function is found by adding its divisors' outputs. For example, $s(27) = \phi(1) + \phi(3) + \phi(9) + \phi(27)$. You just need a little patience to find the sweet child of ϕ.

n	$\phi(n)$	$s(n)$
1	1	1
p		
p^2		
p^3		
p^4		

Darryl's favorite H*R character is the Poopsmith (no surprise). What's yours?

17. Sketch a proof that ϕ is a multiplicative function.

18. Use geometric reasoning, rather than the algebraic formula given today, to find the diameter of a circle placed in the "gap" between two circles of diameter $\frac{1}{a}$ and $\frac{1}{b}$.

The ending rule is simpler using $\frac{1}{a}$ and $\frac{1}{b}$, but you might prefer to use other variables in the short term. There are multiple methods. . . no real easy way out, though.

19. Consider the circles from today's problem, specifically the set of *all* the small circles (starting with the one that has diameter $\frac{1}{4}$. Give an argument that states that the total area of *all* these circles is finite.

But there are an infinite number of circles! And the geometric series rule is no help! Life's tough.

20. Find all solutions to each of the following.
(a) $x^4 = 1$ in mod 5
(b) $x^6 = 1$ in mod 7
(c) $x^{24} = 1$ in mod 35
(d) $x^8 = 1$ in mod 15

21. Here's a weird function: Let $f(n) = 1$ if n can be written as the sum of two squares, and $f(n) = 0$ if it can't be done.
(a) Tabulate f from 1 to 24.
(b) Is f multiplicative?
(c) How does the child of f behave?

The child of f likes to throw tantrums whenever it can't find its favorite square blocks.

22. Let $f(n) = 1$ if n is a perfect square mod 63, and 0 if not. Is f multiplicative?

Tough Stuff

23. Use the circles from today's problems to say something interesting about the infinite sum

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots$$

24. Relate the previous problem to the behavior of function d from Problem 12.

25. Prove that a function is multiplicative if and only if its child is multiplicative.

26. Find an identity that shows that if $m = a^2 + b^2 + c^2 + d^2$ and $n = e^2 + f^2 + g^2 + h^2$, then mn can also be written as the sum of four squares.

Did you know that m and n are the norms of quaternions with integer coordinates? How does that make you feel?