Today we take it back to My Old School, with everything older than (anyone on) the hills.

14 *SS2P 2009: Rated PG*

Important Stuff

SS2P: Strictly Sums To Products!

PROBLEM

The s_2 function is a bit of a mess. It gets up, gets down, it's zero a lot, and when it's not zero it's almost always a multiple of 4. One way to deal with a bizarre function like this one is to turn it into a running average.

So, any way you like, compute the average value of $s_2(n)$ when

(t) n goes from 1 to 25
(e) n goes from 1 to 49
(r) n goes from 1 to 75
(i) n goes from 1 to 108

You may find the handout from Day helpful, mostly.

This function would lose its mind in Detroit Rock City.

To everything, turn, turn, turn it into a running average.

Dang, this number seems to have disappeared faster than the guy who did "Spirit in the Sky".

1. Sandy believes that

$$\begin{aligned}1+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\cdots = &\left(1+\frac{1}{2^s}+\frac{1}{4^s}+\frac{1}{8^s}+\cdots\right)\\ &\cdot\left(1+\frac{1}{3^s}+\frac{1}{9^s}+\frac{1}{27^s}+\cdots\right)\\ &\cdot\left(1+\frac{1}{5^s}+\frac{1}{25^s}+\frac{1}{5^{3s}}+\cdots\right)\\ &\cdot\left(1+\frac{1}{7^s}+\frac{1}{7^{2s}}+\frac{1}{7^{3s}}+\cdots\right)\cdots\\ &\cdot\left(1+\frac{1}{p^s}+\frac{1}{p^{2s}}+\frac{1}{p^{3s}}+\cdots\right)\cdots\end{aligned}$$

Is she right? Why or why not?

Sandy is all right now. They're bad references, but at least they're free.

2. Each of the infinite sums above, like

$$1+\frac{1}{2^s}+\frac{1}{4^s}+\frac{1}{8^s}+\cdots,$$

is a geometric series. Find the value of each geometric series. Rewrite the messy equation above as simply as you can.

3. **Warning:** Notation ahead!

$$\sum_{n=1}^{5} f(n) = f(1) + f(2) + f(3) + f(4) + f(5)$$

$$\sum_{n=1}^{\infty} f(n) = f(1) + f(2) + f(3) + f(4) + f(5) + \cdots$$

$$\prod_{n=1}^{5} f(n) = f(1) \cdot f(2) \cdot f(3) \cdot f(4) \cdot f(5)$$

$$\prod_{p} f(p) = f(2) \cdot f(3) \cdot f(5) \cdot f(7) \cdot f(11) \cdots$$

Write the messy equation from the last problem using as-clean-as-you-can notation, then celebrate by marching in a parade.

A parade? What? Oh, right, we had one of those. But today it's a classic rock Hit Parade!

4. Let $s = 1$ in the messy equation. What happens? Use this to prove that there must be infinitely many prime numbers.

5. Calculate enough terms of this infinite product so that you can identify what the answer is.

$$\left(1 - \frac{1}{2^s}\right)\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{5^s}\right)\left(1 - \frac{1}{7^s}\right)\cdots\left(1 - \frac{1}{p^s}\right)\cdots$$

$$= \frac{?}{1^s} + \frac{?}{2^s} + \frac{?}{3^s} + \frac{?}{4^s} + \frac{?}{5^s} + \frac{?}{6^s} + \cdots$$

Be strategic about expansion; what kinds of terms will you get?

Curses! FOIL'd again! I would've gotten away with it too, if it weren't for those meddling kids.

6. Multiply this out. What happens?

$$\left(\frac{1}{1^s} + \frac{-1}{2^s} + \frac{-1}{3^s} + \frac{0}{4^s} + \frac{-1}{5^s} + \frac{1}{6^s} + \cdots + \frac{\mu(n)}{n^2} + \cdots\right)\cdot$$

$$\left(\frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \cdots + \frac{1}{n^2} + \cdots\right) = \text{hmmmm}$$

7. Find the result of this product. Use the last two problems, buddy/buddette!

$$\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{9}\right)\left(1 - \frac{1}{25}\right)\left(1 - \frac{1}{49}\right)\cdots\left(1 - \frac{1}{p^2}\right)\cdots$$

Review Your Stuff

The final day of this course is mostly taken up by review problems. So we think it would be a good idea for groups to form some summarizing questions that come out of whatever you might find valuable in this course. So, we want your table to write two problems on any subject that has cropped up in the course.

Here are some topics you might consider writing problems about.

Multiplicative functions	σ, τ, ϕ, μ
Modular arithmetic	Parents and children
Circles and summations	Power series
Convergence and divergence	Old school rap

The goal is to create a review whose problems get at the fact that we've come a long, long way in three weeks. The problems should help others synthesize their learning of the aforementioned topics.

So, don't write any stumpers; consider yourself writing two problems that could both fit into "Important Stuff." If your table wants to write more than two, thats fine, and the extra questions can be a little more "Neat" or "Tough." We reserve the right to combine, edit, change, ignore, or otherwise mangle your problems. And it's ok to be funny, as long as it doesnt get in the way of the math.

Neat Stuff

8. What does this equal? Give a justification.

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} s_2(k)}{n}$$

9. Imagine two dice with an infinite number of sides, labeled 1 to ... um, yeah. Two of these presumably spherical dice are rolled, and a result is calculated: the greatest common divisor of the two numbers rolled.

(a) Try this a few times. Randomly pick ten pairs of five- or six-digit numbers, then calculate the greatest common divisor of each pair. Anything surprising?

(b) Explain why it's exactly 4 times more likely for the result to be 1 than for it to be 2.

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(c) How many times more likely is it for the result to be 1 than 3?

(d) Find the exact probability that the result is 1.

10. Use a result from this week to prove that no positive integer can have more factors that are "3 mod 4" than factors that are "1 mod 4".

11. (a) What's the formula for the area of a circle?

(b) What's the formula for the volume of a sphere?

(c) What's the formula for the, uh, hypervolume of a four-dimensional, uh, hypersphere?

Either use calculus or look this up. Easy choice, right?? We hope the answer is still surprising.

12. In today's box you calculated the long-term average value of s_2. Try again with s_4, and see if you find anything interesting.

13. Here's a grid of n^2 fractions:

$$\begin{matrix} \frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} \\ \frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \cdots & \frac{2}{n} \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{n}{1} & \frac{n}{2} & \frac{n}{3} & \cdots & \frac{n}{n} \end{matrix}$$

As n grows, what proportion of the fractions are in lowest terms? $\frac{6}{5}$ is in lowest terms, but $\frac{6}{4}$ isn't.

14. Show that

$$\sum_{d|n} |\mu(d)| = 2^{\text{number of distinct primes dividing } n}$$

15. You are the first contestant on the "Showcase Showdown" of Price is Right, and on your first spin you get 65 cents. Are you more likely to win by spinning again and risking going over \$1.00, or by staying on 65 cents? Rigorously defend your logic.

Tough Stuff

16. What does this infinite product equal?

$$\left(1+\frac{1}{4}\right)\left(1+\frac{1}{9}\right)\left(1+\frac{1}{25}\right)\left(1+\frac{1}{49}\right)\cdots\left(1+\frac{1}{p^2}\right)\cdots$$

17. Prove that no positive integer can have more factors that are "2 mod 3" than factors that are "1 mod 3". Generalize to other mods... if possible.

18. Find a way to generate all of the Pythagorean triples in which *the two leg lengths* are one away from each other. One example is 21, 20, 29.

19. Solve the continuous version of the "Showcase Showdown" problem above, where numbers are picked continuously from 0 to 1 instead of discretely by increments of 0.05. Find the cutoff number n where it's correct to stay when you get more than n on the first try, and to go again with less than n.

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