Thanks everyone! Enjoy the rest of your summer.

15 *Last Call*

Supremely Unimportant Stuff

0. Let $f(n)$ be the height of the day number on Day n's problem set. Is f linear, quadratic, exponential, or something else? If $f(1) = 1$ unit, is f multiplicative?

Ask Darryl why the 15 isn't large enough to take up an entire page.

Your Stuff

4. Notice that

$$\sigma(8) = \sigma(7) + \sigma(6) - \sigma(3) - \sigma(1)$$

Is it also true that $\sigma(9) = \sigma(8) + \sigma(7) - \sigma(4) - \sigma(2)$? How long does this last? Can you "fix it" when it breaks?

Hm, spoken like someone from the Number Theory working group, so a special note for them: I hear $\mathbb{Z}[\sqrt{-163}]$ has unique prime factorization! What's up with that?

8. Find the solutions to
(a) $x^3 = 1 \bmod 7$
(b) $x^3 = 1 \bmod 13$
(c) $x^3 = 1 \bmod 19$
What do you notice in each case?

Hm, you could use thes to solve $x^3 = 1 \bmod 1729$, which is a very interesting cab number!

10. Find all solutions to $x^2 = 4$ in mod 145 using the factorization $145 = 5 \cdot 29$.

0. Find all solutions to $x^3 - x = 4$ in mod 170.

12. Why does the array model work for $\phi(21)$ but doesn't work for $\phi(20)$?

$$\phi(21) = (3-1)\cdot(7-1) \qquad \phi(20) \neq (4-1)\cdot(5-1)$$

	1	6
1	1	6
2	2	12

↑ $\phi(21)$

	1	4
1	1	4
3	3	12

↑ $\neq \phi(20)$

Thanks everyone! Enjoy the rest of your summer.

Last Call

9. Complete this table giving numbers that have one answer in mod 12, and a second answer in mod 7.

number mod 7												
6							6					
5	12					5						
4					4							11
3				3							10	
2			2							9		
1		1							8			
0	0							7				
	0	1	2	3	4	5	6	7	8	9	10	11

number mod 12

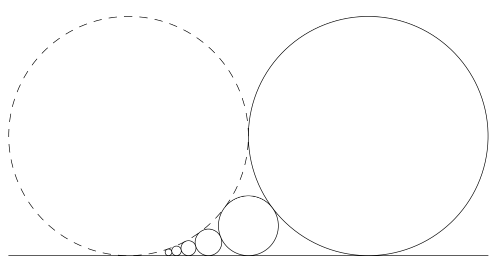
What number between 0 and 84 is 1 in mod 12 and 0 in mod 7? What number is 0 in mod 12 and 1 in mod 7?

9. Repeat the above problem for mod 8 and mod 12. What happens? Is there a number that is 1 in mod 8 and 0 in mod 12?

Commentary from Table 9: "Art is funny." Is he a clown? Does he amuse you? (Yes.)

4. Consider the infinite sequence of circles from Day 8's handout. Find the total circumference of the circles (ignore the dotted circle, whose name is Art).

As a reminder, the sum of the diameters of these circles ended up being this crazy number, $\frac{\pi^2}{6}$.



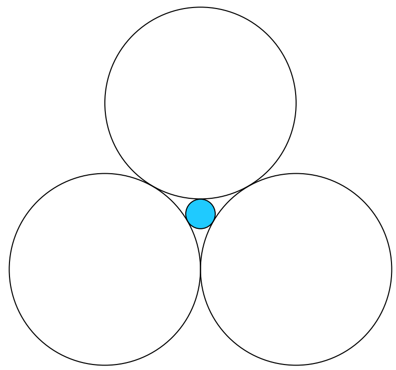
4. Consider the sequence of circles from Day 9, the ones that go left-right-left-right. For each circle, find the x-coordinate of its center. As more and more circles are constructed, what happens to the centers? Specifically, do these circles' centers head toward the point $(\frac{6}{\pi^2}, 0)$?

The circles go left, right, left, right, like an N*SYNC dance number. Bye bye bye PCMI!

Thanks everyone! Enjoy the rest of your summer.

12. The diagram below represents Felipe, his friend Jennifer and his other friend Jennifer. All three of them are shown as circles with diameter 1 that are tangent to each other. Now plop a circle in the middle and call him Adnan (the shaded circle below)!

And lo, a PLOP was heard through the land, and they called him Adnan.



Descartes found this (totally bodacious) formula relating the four diameters; as before, each variable is the *reciprocal* of the real diameter:

$$(a+b+c+d)^2 = 2(a^2+b^2+c^2+d^2)$$

Amazing how simple this formula is, and how similar it is to the case of using a tangent line. Hey wait, what happens when you let $d = 0$? What's up with that?

(a) What is Adnan's diameter?

(b) Now do it again, plopping yet another circle called Marty, between Adnan and the Jennifers. What is Marty's diameter?

(c) As more circles are plopped, what happens to their diameters? Do the diameters form a recognizable sequence?

Marty would also like to hang out with Felipe, but can't reach across Adnan.

2. We've been computing the ways to write any integer as the sum of two squares. As n increases, does the number of ways generally increase? Is there a way to predict the density of the numbers that can be written as the sum of two squares?

You could define a new function that gives 1 when the number *can* be written as the sum of two squares, and 0 when it can't. There's a nice way to do this with power series, but it's not easy to find.

2. In the box from Day 14, as n increases, what happens to the average value of $s_2(n)$? How is this *physically* and/or geometrically related to the handout from Day 11, which gave the $x^2 + y^2$ value at each location (x, y)?

Thanks everyone! Enjoy the rest of your summer.

3. Fill in the following table.

$\zeta(s)^n$	Calculation	Result
$\zeta(s)^{-1}$		
$\zeta(s)^0$		
$\zeta(s)^1$	$\frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\cdots$	$\frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\cdots$
$\zeta(s)^2$	$\left(\frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\cdots\right)^2$	$\frac{\ }{1^s}+\frac{\ }{2^s}+\frac{\ }{3^s}+\frac{\ }{4^s}+\cdots$
$\zeta(s)^3$		

That crazy squiggle is a "zeta", not a "weird C/S lookin' thing". Others may call $\zeta(s)$ the "Riemann zeta function" but you say it's just a friend. What does multiplying by $\zeta(s)$ seem to do to a function? Dividing?

Make a connection between what you see here and one of the "problems in the box".

3. Prove that if f is multiplicative, then its child g must also be multiplicative.

If Chucky and his bride have a child, can the child reproduce? [Seriously, a "Bride of Chucky" gag? It wasn't us.]

10. Given a function, how do you obtain the child? parent? grandchild? grandparent? Take a function and calculate its grandparent and grandchild functions, then multiply those together after writing them as $\frac{?}{1^s}+\frac{?}{2^s}+\cdots$. What happens?! Wow!

7. You know that a multiplicative function satisfies $f(xy) = f(x)f(y)$. Or we hope you do! Find a function for which each of these is true for any choices of the variables.

(k) $\texttt{traci}(x+y) = \texttt{traci}(x) + \texttt{traci}(y)$

(e) $\texttt{randy}(x+y) = \texttt{randy}(x) \cdot \texttt{randy}(y)$

(n) $\texttt{aaron}(xy) = \texttt{aaron}(x) + \texttt{aaron}(y)$

(t) $\texttt{chris}(x+y) = (\texttt{chris}(x))^y$

We apologize for the bizarre function names, which serve no purpose other than for us to be Equal Opportunity Name-Droppers.

7. Given any integer $n > 2$, describe how to find a sequence of n consecutive *non*-primes. Does this contradict the earlier finding that there are an infinite number of primes?

12. Is there a "family tree" of all the functions we've studied these three weeks? If so, what does it look like? It may help to think that functions in the form $f(n) = x^n$ are all "related" though this isn't a parent-child relationship.

Maybe they're married or in the same high school class or something, we're really stretching the analogy for no good reason here.

Thanks everyone! Enjoy the rest of your summer.

1. Show that $x^2 + y^2 = 2003$ has no integer solutions.

(k) If you had easy access to complete tables for all functions we have studied, which would you go to to show this result immediately?

(a) What are the possible values of x^2 in mod 4?

(t) Based on this, what can you say about the possible values of $x^2 + y^2$ in mod 4?

(h) Can 2009 be written as the sum of two squares? If so, find one way to do it.

(y) When is the next year that *cannot* be written as the sum of two squares? What will be the top song that year?

Clearly it will be the re-release of "Cars That Go Boom".

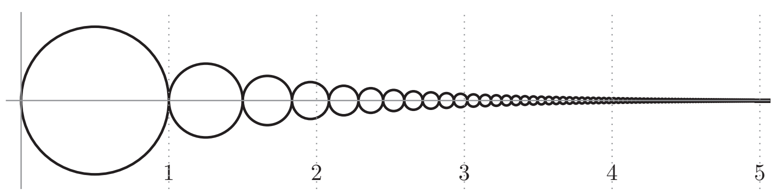
8. How many lattice points are on the graph of $x^2 + y^2 = 3530$? What are they?

Eww; "What are they?" is much tougher than "How many" for this kind of problem.

Our Stuff

1. Here are some circles. Hey, they sort of look like a torus line in a parade, and maybe those two leading the line are Amelia and Carol. Each circle is constructed so that the nth circle has diameter is $\frac{1}{n}$. Find the total circumference and area of *all* the circles built this way.

A Torus Line: great math song, or greatest math song? You decide.



2. Consider m and n, relatively prime integers. Suppose a is 0 in mod m and 1 in mod n, and b is 1 in mod m and 0 in mod n. If $0 \le a, b \le mn$, show that $b = mn - a + 1$.

3. Define a function λ that gives $\lambda(n) = 1$ if n has an even number of prime factors and $\lambda(n) = -1$ if n has an odd number of prime factors. This function takes into account the number of times that a prime factor is repeated. So, for example, $\lambda(1) = 1$ because 1 has zero prime factors, $\lambda(2) = -1$ because 2 has one prime factor (namely, 2). The value of $\lambda(4) = 1$ because $4 = 2^2$ so it has the prime factor 2 appearing twice. Fill in this table with the values of λ, its child `louis` and its parent `mary`.

Now we've gone from cows to sheep. The moo function and this lamb-duh function seem pretty similar.

Thanks everyone! Enjoy the rest of your summer.

Last Call

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
mary(n)	1														
$\lambda(n)$	1	-1	-1	1											
louis(n)	1														

n	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
mary(n)	1														
$\lambda(n)$	1														
louis(n)	1														

4. Prove that if a, b, c are positive integers and

$$(a+b+c+d)^2 = 2(a^2+b^2+c^2+d^2)$$

then d must also be an integer. What does this imply about these circle packing diagrams?

Warning: this statement might not be true! And that last statement definitely isn't true. But the previous two statements are both lies.

Tough Stuff

5. Define $\texttt{lori}(n) = \sum_{k=1}^{n} \lambda(k)$. Is $\texttt{lori}(n) \le 0$ for all $n > 1$? Prove or find a counter-example.

6. Two days ago we showed that the equation

$$x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\cdots = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{7!} + \cdots$$

leads to a proof that $1 + \frac{1}{4} + \frac{1}{9} + \cdots = \frac{\pi^2}{6}$. What about the x^5 term? Is there some formula involving 120 in there?

7. Calculate $\displaystyle\sum_{n=2}^{\infty}\sum_{m=1}^{n-1} \frac{1}{(mn)^2}$.

8. Calculate $\displaystyle\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{(mn)^2}$. It'll be related to $\frac{\pi^2}{6}$.

9. Use the previous two facts and a lovely grid to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

10. Use a similar setup to find the exact value of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^6}$.

11. Explain why $e^{\pi\sqrt{163}}$ is an integer. Amazing!