6 *Children of M(n)*

Hey, welcome to the class. We know you'll learn a lot of mathematics here—maybe some new tricks, maybe some new perspectives on things with which you're already familiar. A few things you should know about how the class is organized:

This is reprinted from Day 1, since we have many new people in the room to start Week 2. Hope you had a great weekend! Also, we hope you're a fan of Cuarón or the title probably makes no sense.

- **Don't worry about answering all the questions.** If you're answering every question, we haven't written the problem sets correctly.
- **Don't worry about getting to a certain problem number.** Some participants have been known to spend the entire session working on one problem (and perhaps a few of its extensions or consequences).
- **Stop and smell the roses.** Getting the correct answer to a question is not a be-all and end-all in this course. How does the question relate to others you've encountered? How did others at your table think about this question?
- **Respect everyone's views.** Remember that you have something to learn from everyone else. Remember that everyone works at a different pace.
- **Learn from others.** Give everyone the chance to discover, and look to your tablemates for new perspectives on problems. Resist the temptation to tell others the answers if they aren't ready to hear them yet. If you think it's a good time to teach your tablemates about dividing by the zeta function, think again: the problems should lead to the appropriate mathematics rather than requiring it. The same goes for technology: the problems should lead to the appropriate use of technology rather than requiring it.
- **Each day has its Stuff.** There are problem categories: Important Stuff, Neat Stuff, Tough Stuff, and maybe other stuff sometimes. Check out Important Stuff first. All the mathematics that is central to the course can be found and developed in the Important Stuff. *That's* why it's Important Stuff. Everything else is just neat or tough. If you didn't get through the Important Stuff, we noticed... and the important questions recycle. Each problem set is based on what happened before, either in problems or in class discussions.

We promise to add more unsolved problems to the two that have already appeared on the problem sets.

Resist! Resist, we say!

Alternate title for today: One Function Per Child

Important Stuff

PROBLEM

Last week we learned how functions make babies. The general rule is that

$$\text{child}(n) = \text{parent}(\text{all divisors of } n) \text{ added together}$$

Andy said it had something to do with a stork and an input-output table...

Today's amazing parent function is $m(n) = 1$. Yup, you got it. It's a function that always gives you the number 1. It's not a particularly interesting function but it has a fabulous family!

What is $m(7)$? What is $m(m(7))$? What is $m(Ana)$? You get the idea.

Let t be the child of m and let u be the child of t. This means that

$$\begin{aligned} t(n) &= m(\text{all divisors of } n) \text{ added together} \\ u(n) &= t(\text{all divisors of } n) \text{ added together} \\ t(1) &= m(1) = 1 \\ t(15) &= m(1) + m(3) + m(5) + m(15) = 4 \\ u(16) &= t(1) + t(2) + t(4) + t(8) + t(16) = ?? \end{aligned}$$

Fill in this table with the values of $t(n)$ and $u(n)$.

n	1	2	3	4	5	6	7	8	9	10	11	12
$m(n)$	1											
$t(n)$	1											
$u(n)$	1											

n	13	14	15	16	17	18	19	20	21	22	23	24
$m(n)$												
$t(n)$			4									
$u(n)$												

1. Is t multiplicative? How about u? m?

Um?... how about *you*? A function is *multiplicative* if $f(ab) = f(a) \cdot f(b)$ whenever a and b don't have a common factor greater than 1.

2. You may have noticed in the above problem that primes play a large role in determining the values of these functions. Complete this shorter table for each function.

n	$m(n)$	$t(n)$	$u(n)$
1			
p			
p^2			
p^3			
p^4			

3. Use the tables from Problem 2 to quickly compute $m(420)$, $t(420)$, and $u(420)$.

420, eh? Hopefully this problem won't leave you dazed and confused. Sorry, that was a half-baked reference.

4. (a) If n is one more than a multiple of 3 and one more than a multiple of 8, what's the most you can say about the value of n?

(b) If n is 2 more than a multiple of 3 and 7 more than a multiple of 8, what's the most you can say about the value of n?

5. On Friday, Assegid mentioned that many perfect squares appeared to be one more than a multiple of 24. For what values of n, from $1 \le n \le 48$, is n^2 one more than a multiple of 24?

A calculator might be handy; those squares get big. Or use the "double plus 1" rule:

$$15^2 = 14^2 + (2 \cdot 14 + 1)$$

$$16^2 = 15^2 + (2 \cdot 15 + 1)$$

6. Find all solutions to each of the following.

(a) $x^2 = 1$ in mod 3
(b) $x^2 = 1$ in mod 8
(c) $x^2 = 1$ in mod 24
(d) $x^2 = 1$ in mod 72

7. Last week you worked with the ϕ function, which counts how many numbers between 1 and n have no common factors with n (larger than 1). For example, $\phi(15) = 8$ because the numbers 1, 2, 4, 7, 8, 11, 13, and 14 all have no common factors with 15. (Note that $\phi(1) = 1$, not 0.)

(a) List all the numbers that make up $\phi(1), \phi(3), \phi(7)$, and $\phi(21)$.

(b) What is $\phi(1) + \phi(3) + \phi(7) + \phi(21)$?

(c) Simplify this expression as much as possible:

$$1 + (p-1) + (q-1) + (p-1)(q-1)$$

And by "you" we mean *you*, Lalit. Okay, maybe we mean everyone who was here last week!

Neat Stuff

8. Complete this table for some "fan favorite" functions from Week 1. $\sigma(n)$ is the sum of the factors of n, $a(n) = \frac{\sigma(n)}{n}$, and $\phi(n)$ is given in Problem 7.

n	$\sigma(n)$	$a(n)$	$\phi(n)$
1			
p			
p^2			
p^3			
p^4			

Answers here are in terms of p, so you might consider testing some different choices of p first. But not $p = 4\ldots$

9. If v is the child of u, what would its column in the table from Problem 2 look like? Compute $v(420)$.

Another 420?! Hopefully this time it's just a reference to blackbirds baked in a pie.

10. (a) If x is 1 mod 3, 0 mod 5, and 0 mod 7, what is it in mod 105?
(b) If y is 0 mod 3, 1 mod 5, and 0 mod 7, what is it in mod 105?
(c) If z is 0 mod 3, 0 mod 5, and 1 mod 7, what is it in mod 105?
(d) Compute $2x + 3y + 4z$ in mod 105.

11. If x is 2 mod 3, 3 mod 5, and 4 mod 7, what is it in mod 105?

12. Find all solutions to each of the following. Don't use a calculator unless you feel really, really compelled to.

(a) $x^2 = 1$ in mod 3
(b) $x^2 = 1$ in mod 5
(c) $x^2 = 1$ in mod 7
(d) $x^2 = 1$ in mod 15
(e) $x^2 = 1$ in mod 21
(f) $x^2 = 1$ in mod 35
(g) $x^2 = 1$ in mod 105
(h) $x^2 = 1$ in mod 420

13. Find all solutions to each of the following. See previous rant about calculator usage.

Vicki claims the result in part (a) is very helpful in some later parts. And she's right!

(a) $x^2 = 2$ in mod 3
(b) $x^2 = 2$ in mod 7
(c) $x^2 = 2$ in mod 17
(d) $x^2 = 2$ in mod 21
(e) $x^2 = 2$ in mod 51
(f) $x^2 = 2$ in mod 119
(g) $x^2 = 2$ in mod 357... which is $3 \times 7 \times 17$

14. Prove that if f is multiplicative, then either $f(1) = 1$ or $f(n) = 0$ all the time.

15. Prove that if f and g are multiplicative functions, then $h = fg$, the product, is also multiplicative.

16. Find a value of n for which $x^2 = 2$ has exactly 8 solutions in mod n.

It's over there! Scavenger hunt bonus: actually *find* this number.

17. Define $c(x) = \frac{\phi(x)}{x}$.
(a) Explain why $c(x) < 1$ as long as $x > 1$.
(b) Find a value for which $c(x) < 0.1$ or show there is no such x.
(c) Find the minimum possible value of $c(x)$.

18. We said $m(n) = 1$ had a fabulous family, but only showed the children!

(a) Find the parent of m; that is, a function d so that m is the child of d.

(b) Find z, the parent of d.

Think of the children!! Won't *someone* think of the children?!

19. Prove that if f is multiplicative, then so is $\frac{1}{f}$, the reciprocal, as long as $f(n) \neq 0$ for any n.

20. Prove that the child of ϕ is the identity $r(n) = n$.

21. Find a rule or give a formula for each function.

(a) $f(m)$ is the number of solutions to $x^2 = x$ in mod m.

(b) $g_1(m)$ is the number of solutions to $xy = 1$ in mod m.

(c) $g_2(m)$ is the number of solutions to $x^2 - y^2 = 1$ in mod m.

(d) $h_1(m)$ is the number of solutions to $x^2 - y^2 = 0$ in mod m.

(e) $h_2(m)$ is the number of solutions to $x^2 + y^2 = 0$ in mod m.

Tough Stuff

22. Under what conditions is 2 a perfect square in mod m? Be careful about composite m...

Go away! Stop working on the Tough Stuff, especially after hours!

23. Prove that if g is the child of f, and f is multiplicative, then g is multiplicative.

24. Prove that if g is the child of f, and f is *not* multiplicative, then g is *not* multiplicative.

25. Can a nonzero multiplicative function be its own ancestor (allowing for more than one generation)?

26. Let $s_4(n)$ be the number of ways to write n as the sum of four squares: $n = a^2 + b^2 + c^2 + d^2$ where a, b, c, d are integers (perhaps zero or negative). Starting with $s_4(1) = 8$, tabulate s_4 from 1 to 24. Is s_4 multiplicative? Fix it!

27. This identity shows that if m and n can be written as the sum of two squares, then mn can be written as the sum of two squares:

$$(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2$$

Find an identity that shows the same fact when m and n are each written as the sum of four squares.

Clearly, a problem like this requires the use of quaternions. What are those? *I* don't know, but ask MIB agents *J* and *K*.

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7 *Sweet Child O' Mine*

Important Stuff

We were going to call today's set "Children of the Corn" for linear algebra fanatics, a nod to kernel reference. That would've been a-maize-ing.

PROBLEM

Pick one or more of the following equations. Coordinate with those at your table so that not everyone picks the same equation.

- $4x = 8$
- $x^2 + x = 2$
- $x^2 = 1$
- $x^2 = 2x$
- $x^2 + 2x = 3$
- $x^3 + 2x + 3 = 0$

It's time for some x-coordination!

On the array of numbers from 0 to 59 below...

1. Draw a circle around all of the numbers that are solutions to your equation in mod 5. That means if you circled 2, you should also circle 7, 12, 17...
2. Draw an X through all of the numbers that are solutions to your equation in mod 12. That means if you X'ed 2, you should also X 14, 26...
3. Find all solutions to your equation in mod 60.

Instead of using circles and X's, you could also use markers! (Darryl: Polars!) Yes, he actually said this.

0	1	2	3	4	5	6	7	8	9	10	11
12	13	14	15	16	17	18	19	20	21	22	23
24	25	26	27	28	29	30	31	32	33	34	35
36	37	38	39	40	41	42	43	44	45	46	47
48	49	50	51	52	53	54	55	56	57	58	59

Let $f(n)$ be the number of solutions to your equation in mod n, remembering that the actual numbers in mod n go from 0 to $n - 1$. What is $f(5)$? What is $f(12)$? What is $f(60)$?

Make sure $f(5)$ isn't larger than 5...that would be unpossible!

1. A number is 3 more than a multiple of 5, and 7 more than a multiple of 12. Find some possible values of this number.

2. $\phi(n)$ counts the number of values between 1 and n that are relatively prime to n.

(a) What is $\phi(5)$? What is $\phi(12)$?

(b) If a number shares no common factors with 5 *and* no common factors with 12, what can you say about it?

(c) Use the method from today's box problem, or the method presented yesterday, to find all the numbers that are relatively prime to 60.

1	2	3	4	5	6	7	8	9	10	11	12
13	14	15	16	17	18	19	20	21	22	23	24
25	26	27	28	29	30	31	32	33	34	35	36
37	38	39	40	41	42	43	44	45	46	47	48
49	50	51	52	53	54	55	56	57	58	59	60

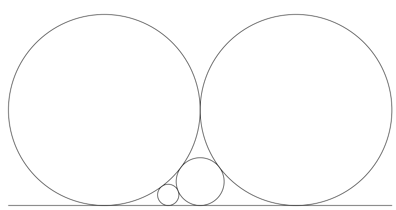
(d) What is $\phi(60)$?

Two numbers are *relatively prime* if they share no common factors greater than 1. It's just a quick way of saying that longer phrase.

3. Let $P(n)$ be defined as the number of solutions to $xy = 1$ in mod n. What is $P(5)$? What is $P(12)$? What is $P(60)$?

Actually tracking down all the solutions making up $P(60)$ might take a while, so perhaps you could find a faster way!

4. Two circles with diameter 1 are tangent to the x-axis, and mutually tangent at the point $(\frac{1}{2}, \frac{1}{2})$. Two smaller circles are packed in as shown below so they are tangent to the axis and the other circles. Use the overlaid grid on the handout to find the diameters and the coordinates of the centers of each of these smaller circles.



Lots of circles and X's today. Matt and Debra have abandoned the problem set in favor of tic-tac-toe.

5. Fill in these two grids; one is the set of all possible sums when rolling two dice, and the other is the piece-by-piece expansion of $(x + x^2 + x^3 + x^4 + x^5 + x^6)^2$.

+	1	2	3	4	5	6
1						
2			5			
3						
4						
5		7				
6						12

$\times$	x	x^2	x^3	x^4	x^5	x^6
x						
x^2			x^5			
x^3						
x^4						
x^5		x^7				
x^6						x^{12}

Cool, huh?

Ahem! In Gema's home state of Texas, these are called *number cubes*! Get it right!

6. Use technology to build a histogram for the number of ways (or, if you prefer, the probability) to roll each possible sum with four dice, from 4 to 24.

I said *number cubes* consarnit! Get a calculator screen (HOME icon), define $p(x)$ to be some specific interesting polynomial ("define $p(x) = \cdots$"), then tell the calculator:

expand$(p(x)^4)$

Neat Stuff

7. Define $c(n) = \frac{\phi(n)}{n}$.

(a) Make a table for c from 1 to 36, using exact fractional answers.

(b) Complete this table for $c(n)$ looking at powers of primes.

n	$c(n)$
1	
p	
p^2	
p^3	
p^4	

(c) Do you think it is possible for $c(n)$ to be less than 0.1? Explain.

8. Let $d(n) = a(n) \cdot c(n)$, where $a(n)$ is the sum of the reciprocals of the factors of n (see Day 2) and $c(n)$ is defined in Problem 7.

(a) Compute $d(n)$ using any method from 1 to 36, giving decimal answers to four places.

(b) Does d seem to be multiplicative?

(c) Does d seem to have a maximum value? A minimum?

We named this function after David. Or, maybe it was just the next available letter in the alphabet, we forget which...

9. I'd like to solve these, Pat.

Is there an x in the puzzle?

(a) $x^2 + x = 2$ in mod 3

(b) $x^2 + x = 2$ in mod 5

(c) $x^2 + x = 2$ in mod 7

(d) $x^2 + x = 2$ in mod 15

(e) $x^2 + x = 2$ in mod 21

(f) $x^2 + x = 2$ in mod 35

(g) $x^2 + x = 2$ in mod 105

10. (a) If x is 1 mod 3, 0 mod 5, and 0 mod 11, what is it in mod 165?

(b) If y is 0 mod 3, 1 mod 5, and 0 mod 11, what is it in mod 165?

(c) If z is 0 mod 3, 0 mod 5, and 1 mod 11, what is it in mod 165?

(d) Compute $2x + 4y + 8z$, answering in mod 165.

11. If x is 2 mod 3, 4 mod 5, and 8 mod 11, what is it in mod 165?

12. How many solutions are there to $x^2 = 1$ in mod 165?

13. Which of the following functions are multiplicative?
(a) $b(n) = \frac{1}{n}$
(b) $\sigma_2(n)$, the sum of the squares of the divisors of n
(c) $\chi(n) = \begin{cases} 1 & \text{if } n = 4k+1 \text{ for some positive integer } k \\ -1 & \text{if } n = 4k-1 \text{ for some positive integer } k \\ 0 & \text{if } n \text{ is even} \end{cases}$

That χ function (pronounced "kai" like the Cobra dojo) is pretty wacky. A little *too* wacky if you ask me.

14. Katya says that you can identify a multiplicative function just by declaring what it does to powers of primes. For each description, give a simple rule for the function f.
(a) $f(p^k) = k + 1$
(b) $f(p^k) = 1$
(c) $f(p^k) = p^k$
(d) $f(p^k) = 1 + p + p^2 + \cdots + p^k$

15. Complete this table for $\phi(n)$ and its child.

Remember, the *child* of a function is found by adding its divisors' outputs. For example, $s(27) = \phi(1) + \phi(3) + \phi(9) + \phi(27)$. Sweet child o' ϕ!

n	$\phi(n)$	$s(n)$
1	1	1
p		
p^2		
p^3		
p^4		

16. Sketch a proof that ϕ is a multiplicative function. Your work in today's Important Stuff may be helpful.

17. Prove that if f and g are multiplicative functions, then $h = fg$, the product, is also multiplicative.

18. Last Friday, we saw that the child of the $\phi(n)$ function is the identity function. For example, we saw that

$$\phi(1) + \phi(3) + \phi(5) + \phi(15) = 15$$

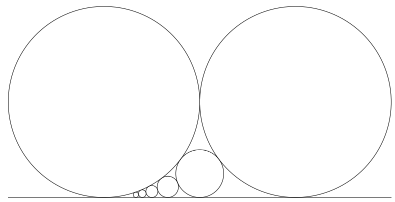
$$\phi(1) + \phi(2) + \phi(7) + \phi(14) = 14$$

Prove that in general,

$$\sum_{d|n} \phi(d) = n$$

Holy notation, Batman! $d|n$ means "d divides n", or that d is a factor of n. This allows the summing to happen only for specific values of d instead of the usual 1, 2, 3, 4...

19. Two circles with diameter 1 are tangent to the x-axis, and mutually tangent at the point $(\frac{1}{2}, \frac{1}{2})$. A big pile of smaller circles are packed in as shown below so they are tangent to the axis and the other circles. Use your mad geometry skills to determine the diameters and the coordinates of the centers of each of these smaller circles.



20. Let $a(n)$ be the sum of the reciprocals of the factors of n. Find a simple rule for the *parent* of this function.

How is a defined? Does this help you find the parent more easily?

21. Yesterday, we said $m(n) = 1$ had a fabulous family, but focused on its children.
(a) Find the parent of m; that is, a function d so that m is the child of d.
(b) Find z, the parent of d.
(c) Describe a general method to find the parent of any multiplicative function.

22. Find the smallest value of $k > 0$ so that each equation has the *maximum possible* number of solutions.
(a) $x^k = 1$ in mod 5
(b) $x^k = 1$ in mod 15
(c) $x^k = 1$ in mod 105

Tough Stuff

23. Find the minimum possible value of $d(n)$, where d is given in Problem 8.

24. Find the maximum possible value of $b(n)$, the sum of the reciprocals of the squares of the divisors of n.

25. Prove that a function is multiplicative if and only if its child is multiplicative.

26. Can a nonzero function be its own ancestor, allowing for more than one generation?

27. Find an identity that shows that if $m = a^2 + b^2 + c^2 + d^2$ and $n = e^2 + f^2 + g^2 + h^2$, then mn can also be written as the sum of four squares.

Clearly, the solution to this problem relies on a deep understanding of the multiplication of quaternions. Um, yeah.

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Best wishes to those suffering from Sine Flu.

8 *Problem Children*

Important Stuff

An important Calendar Event will happen during class today! Do you know what it is?

PROBLEM

We're off to solve the equation $x^3 - 1 = 0$ in mod 63! Here are the steps:

- Find all solutions to $x^3 - 1 = 0$ in mod 7.
- Find all solutions to $x^3 - 1 = 0$ in mod 9.
- Use a grid technique to find all solutions in mod 63.

We're off to solve the 'quation, the wonderful 'quation of mods...

So, off you go; make a big O around any solution to $x^3 - 1 = 0$ in mod 7 (and its equivalent numbers), write an X over any solution to $x^3 - 1 = 0$ in mod 9 (and its equivalent numbers).

0	1	2	3	4	5	6	7	8
9	10	11	12	13	14	15	16	17
18	19	20	21	22	23	24	25	26
27	28	29	30	31	32	33	34	35
36	37	38	39	40	41	42	43	44
45	46	47	48	49	50	51	52	53
54	55	56	57	58	59	60	61	62

XOXO Gossip Girl! What, come on, you don't watch Gossip Girl? Anyway, if 4 is an answer to $x^3 - 1 = 0$ in mod 7, you'd circle 4, 11, 18, 25... A number is a multiple of 9 whenever its digits add up to a multiple of 9.

Let $f(n)$ be the number of solutions to your equation in mod n, remembering that the actual numbers in mod n go from 0 to $n - 1$. What is $f(7)$? What is $f(9)$? What is $f(63)$?

Make sure $f(7)$ isn't larger than 7... that would be bad. Total protonic reversal bad.

1. (a) If a number is a multiple of 63, is it *required* to be a multiple of 7 and a multiple of 9?

(b) If a number is a multiple of 7 and a multiple of 9, is it *required* to be a multiple of 63?

2. (a) If a number is a multiple of 60, is it *required* to be a multiple of 6 and a multiple of 10?

(b) If a number is a multiple of 6 and a multiple of 10, is it *required* to be a multiple of 60?

3. If you didn't get the chance to do yesterday's circle problem, try it now. Use the grid (counting boxes) to find the diameters and centers of the two smaller circles.

Each box is $\frac{1}{36}$ on a side. Why would we pick such a strange scale??

Best wishes to those suffering from Sine Flu.

4. A wicked awesome formula for dealing with yesterday's circle problem was discovered by Descartes, forgotten, then rediscovered in Japan in the early 19th Century.

Apparently, Descartes was from Revere, Massachusetts. Who knew?

Suppose three circles are all mutually tangent, and tangent to the same line. If the *reciprocals* of the circles' diameters are a, b, c then, astoundingly:

$$2(a^2 + b^2 + c^2) = (a + b + c)^2$$

Ooh, symmetric. Don't forget: these are the *reciprocals*. A formula using the actual diameters can be written, but it's not as wickedly awesome.

(a) Let $a = 1$ and $b = 1$. Use the equation above to find both possible values of c.

(b) Use c to determine the diameter of the first small circle.

(c) Let $a = 1$ and $b = 4$. Use the equation to find both possible values of c.

(d) Find the diameter of the second small circle.

(e) Find the diameter of the third small circle. (What should a and b equal?)

(f) Find the diameter of the fourth small circle, and the fifth.

Wait, in part (b) there are two values of c. What does the other value of c give for the diameter? What would a circle with that diameter look like??

5. Look back at yesterday's set and work through Problems 5 and 6 if you haven't already.

6. A rock-paper-scissors player makes their selection randomly. In five games, they picked paper once, rock three times, and scissors once. In how many different ways could this have happened?

Good ole rock, nothin' beats rock. One way is PRRRS, another is RPRSR.

7. Using an Nspire or other algebraic calculator, compute the expansion of $(p + r + s)^5$ to answer the previous question.

Returning participants may prefer to expand $(b + n + c)^5$ for an alternate version of the game. Ask Bill or Darryl.

Neat Stuff

8. (a) If x is 1 in mod 7 and 0 in mod 9, what is it in mod 63?

(b) If y is 0 in mod 7 and 1 in mod 9, what is it in mod 63?

(c) Compute the following nine quantities, giving answers in mod 63:

$$\begin{array}{ccc} x+y & 2x+y & 4x+y \\ x+4y & 2x+4y & 4x+4y \\ x+7y & 2x+7y & 4x+7y \end{array}$$

What do you notice?

What's In The Box?!

9. Describe a method to solve *any* equation in mod 63.

10. Describe a method to predict the number of solutions to any equation in mod 63.

11. Define $c(n) = \frac{\phi(n)}{n}$.
(a) Make a table for c from 1 to 36, using exact fractional answers.
(b) Complete this table for $c(n)$ looking at powers of primes.

n	$c(n)$
1	
p	
p^2	
p^3	
p^4	

Pick a number for p if you're having trouble. *No, not that number!!!* Ahh, just kidding.

(c) Do you think it is possible for $c(n)$ to be less than 0.1? Explain.

12. Let $d(n) = a(n) \cdot c(n)$, where $a(n)$ is the sum of the reciprocals of the factors of n (see Day 2) and $c(n)$ is defined in Problem 11.
(a) Compute $d(n)$ using any method from 1 to 36, giving decimal answers to four places.
(b) Does d seem to have a maximum value? A minimum?

We named this function after Dianna. Or maybe Dawn. Or Darryl?

13. Solve $x^2 + x = 6$ in mod 105.

14. How many solutions are there to $x^2 = 1$ in mod 1155?

15. Adam says that you can identify a multiplicative function just by declaring what it does to powers of primes. Nobody believes him, but it's true! For each description, give a simple rule for the function f.
(a) $f(p^k) = k + 1$
(b) $f(p^k) = 1$
(c) $f(p^k) = p^k$
(d) $f(p^k) = 1 + p + p^2 + \cdots + p^k$

16. Complete this table for $\phi(n)$ and its child.

n	$\phi(n)$	$s(n)$
1	1	1
p		
p^2		
p^3		
p^4		

Remember, the *child* of a function is found by adding its divisors' outputs. For example, $s(27) = \phi(1) + \phi(3) + \phi(9) + \phi(27)$. You just need a little patience to find the sweet child of ϕ.

17. Sketch a proof that ϕ is a multiplicative function.

18. Use geometric reasoning, rather than the algebraic formula given today, to find the diameter of a circle placed in the "gap" between two circles of diameter $\frac{1}{a}$ and $\frac{1}{b}$.

The ending rule is simpler using $\frac{1}{a}$ and $\frac{1}{b}$, but you might prefer to use other variables in the short term. There are multiple methods... no real easy way out, though.

19. Consider the circles from today's problem, specifically the set of *all* the small circles (starting with the one that has diameter $\frac{1}{4}$. Give an argument that states that the total area of *all* these circles is finite.

But there are an infinite number of circles! And the geometric series rule is no help! Life's tough.

20. Find all solutions to each of the following.
(a) $x^4 = 1$ in mod 5
(b) $x^6 = 1$ in mod 7
(c) $x^{24} = 1$ in mod 35
(d) $x^8 = 1$ in mod 15

21. Here's a weird function: Let $f(n) = 1$ if n can be written as the sum of two squares, and $f(n) = 0$ if it can't be done.
(a) Tabulate f from 1 to 24.
(b) Is f multiplicative?
(c) How does the child of f behave?

The child of f likes to throw tantrums whenever it can't find its favorite square blocks.

22. Let $f(n) = 1$ if n is a perfect square mod 63, and 0 if not. Is f multiplicative?

Tough Stuff

23. Use the circles from today's problems to say something interesting about the infinite sum

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots$$

24. Relate the previous problem to the behavior of function d from Problem 12.

25. Prove that a function is multiplicative if and only if its child is multiplicative.

26. Find an identity that shows that if $m = a^2 + b^2 + c^2 + d^2$ and $n = e^2 + f^2 + g^2 + h^2$, then mn can also be written as the sum of four squares.

Did you know that m and n are the norms of quaternions with integer coordinates? How does that make you feel?

Darryl's favorite H*R character is the Poopsmith (no surprise). What's yours?

9 *Circular Reasoning*

Important Stuff

Today's alternate "child" title (child-parent stuff will return soon): Child of Darkness, Child of Light. Does anyone remember that movie? Nope.

PROBLEM

The table below contains the numbers from 0 to 69 arranged so that their remainders after dividing by 7 and 10 are shown in the first column or last row. The numbers from 0 to 10 have already been filled in for you. Fill in the rest of the numbers.

Here's today's Super Password chain: Simpson, food, shape, zodiac, Ford. Got it?

number mod 7

6							6			
5						5				
4					4					
3	10			3						
2			2							9
1		1							8	
0	0							7		
	0	1	2	3	4	5	6	7	8	9

number mod 10

As an example in the grid, the number 10 is 0 mod 10, and it's 3 mod 7, so it shows up with "coordinates" (0,3). Where should 11 be placed next?

1. (a) Find all solutions to $x^2 - x - 2 = 0$ in mod 7.
(b) Find all solutions to $x^2 - x - 2 = 0$ in mod 10.
(c) Use the table in the box above and your answers in mod 7 and mod 10 to quickly find all eight solutions to $x^2 - x - 2 = 0$ in mod 70. Neat, huh?

2. Try making a chart like the one above, using mod 6 and mod 10 instead. What happens when you try to fill in the chart with numbers from 0 to 59?

3. See today's circletastic handout. A path has been drawn from the edge of the largest circle on the right all the way through all the tiny circles. And we mean *all* of them, even

It's circlerific! Today's jokes are not very circlever.

though they keep going like a battery-powered rabbit. The path is straight between centers of consecutive circles.

(a) How long is the piece of the path that goes through the first small circle (in the center of the diagram)?

(b) How long is the piece of the path that goes through the second small circle?

(c) Write an expression (perhaps one with "$\cdots$") that gives the total length of the path, including the diameter running through the largest circle on the right.

Who would win in a hiking contest between Andrew and that bunny? I vote for Andrew.

One with. . . what? Is this a Buddhism reference?

4. A curved path goes from point A to point B, then point C, then point D. Timon decides to walk in a straight line from A to B, then B to C, then C to D. Is Timon's walk longer or shorter than the path? Always? Explain in brief.

It must be a Buddhism reference, if now we're talking about the curved and straight paths.

5. Function $b(n)$ gives the sum of the reciprocals of the *squares* of the divisors of n. Boy, what a mouthful—an example is far better to see:

$$b(15) = 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{225}$$

For each of these, write the entire list of fractions that make up $b(n)$ or the first k terms as listed.

(a) All terms of $b(10)$

(b) All terms of $b(6)$

(c) All terms of $b(24)$

(d) The first 6 terms of $b(60)$

(e) The first 10 terms of $b(2520)$

The divisors of 15 are 1, 3, 5, and 15; so the denominators are $1^2, 3^2, 5^2$, and 15^2.

2520 is a weird choice. What's up with that?

6. (a) Complete this table, where $f(n)$ is the number of solutions to $x^2 = n$ over the integers. Yes, there really are a lot of zeros in this table.

$f(9) = 2$ because $9 = 3^2$ and $9 = (-3)^2$. We count both 3 and -3. Yay! No mods in this problem!

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$f(n)$	1		0							2			

n	13	14	15	16	17	18	19	20	21	22	23	24	25
$f(n)$													

(b) Is f multiplicative? Explain.

(c) Like the "dice polynomial" $x + x^2 + x^3 + x^4 + x^5 + x^6$, write a polynomial that expresses the number of different ways to write numbers as a perfect square. (This polynomial will include the term $2x^9 \ldots$)

For the purpose of this problem, stop at $2x^{25}$, but keep in mind that this polynomial actually goes on forever with higher-degree terms. This didn't happen with the dice polynomial, which stops at the x^6 term.

7. (a) Use an Nspire or any other method to square the polynomial you found in Problem 6c. Complete this table where $f_2(n)$ is the coefficient of x^n in that squared polynomial.

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$f_2(n)$	1		4							4			

n	13	14	15	16	17	18	19	20	21	22	23	24	25
$f_2(n)$	8												

(b) Is f_2 multiplicative? Explain.

8. (a) Find the number of different ways to write 2 as the sum of two squares. Two such ways are $1^2 + (-1)^2$ and $(-1)^2 + 1^2$, but there are two more.

(b) Find the number of different ways to write 7 as the sum of two squares. That was quick!

(c) How many different ways can you write 9?

(d) 13?

(e) 25?

Yes, we mean as the sum of two squares! Writing 9 as $6+3$ or as $\frac{18}{2}$ doesn't count.

Neat Stuff

9. (a) Use the chart from today's box problem to find a, a number that is 1 in mod 10 and 0 in mod 7.

(b) ... and find b, which is 0 in mod 10 and 1 in mod 7.

(c) Calculate $4a + 6b$ in mod 70.

(d) Calculate $7a + 2b$ in mod 70.

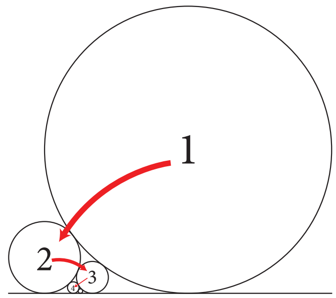
10. Here's an interesting circle packing situation. Start with the right-side circle with diameter 1 (labeled "1" in the diagram on the next page) and the circle of diameter $\frac{1}{4}$ from the usual diagram we've been using (labeled "2"). Now, stuff a tangent circle between those, to the right of the circle with diameter $\frac{1}{4}$. Remember, the formula to find the diameter of new circles is

$$2(a^2 + b^2 + c^2) = (a + b + c)^2$$

For a little light reading, we recommend http://mathforum.org/pcmi/hstp/resources/circlepacking. Thanks, Celeste!

Darryl's favorite H*R character is the Poopsmith (no surprise). What's yours?

In this formula, a, b, c are the reciprocals of the circles' diameters.



(a) What is this small circle's diameter? (It is labeled "3" above.)
(b) Now stuff another circle to the *left* of that (a tiny "4"). What is its diameter? Note: It's not $\frac{1}{16}$!
(c) Now stuff another circle to the *right* of that (uber-teeny "5"). What is its diameter?
(d) Now stuff another circle to the *left* of that (mega-teensy "6"). What is its diameter?
(e) Notice anything?

11. Prove that if f and g are multiplicative functions, then $h = fg$, the product, is also multiplicative. This means that you must prove that $h(ab) = h(a)h(b)$ given that a and b are relatively prime, and you already know that $h(x) = f(x)g(x)$ for any choice of x, and that f and g are multiplicative.

So $h(ab)$ equals what to start off? Blow it to bits!

12. Complete this table for $c(n) = \frac{\phi(n)}{n}$ and $a(n) = \frac{\sigma(n)}{n}$ along with $d(n) = c(n) \cdot a(n)$.

n	$c(n)$	$a(n)$	$d(n) = c(n)a(n)$
1	1	1	1
p	$1 - \dfrac{1}{p}$	$1 + \dfrac{1}{p}$	
p^2			
p^3			

13. Use what you know about the a function to show that for any $k > 0$, no matter how small, there is a value of n for which $c(n) < k$.

14. By hand, find all eight solutions to $x^2 = 3$ in mod 3289.

Okay, we'll be friendly and say that one of the factors of 3289 can be found on any Dr Pepper can.

15. Give an argument that the total area of all the small circles from these circular diagram thingies is finite.

"Thingies" is the technical term for these whozits and whazzamabobs.

16. The centers of the circles from Problem 10 are tending toward a specific point. Find its coordinates.

17. Find the number of different ways to write 25 as the sum of four squares. Two such ways are $0^2 + 0^2 + 5^2 + 0^2$ and $3^2 + 0^2 + (-4)^2 + 0^2$.

Please, please do not write a computer program to do this – think! This question can be answered cleverly using other information from today.

Tough Stuff

18. Let $p(x) = x + x^2 + x^3 + x^4 + x^5 + x^6 + \cdots$, like the dice polynomial, but it goes on forever.

(a) Let $p_2(x) = p(x)^2$, $p_3(x) = p(x)^3$, etc. Write the first few terms of $p_2(x)$, $p_3(x)$, and so forth, until you can describe to someone else what is happening. Then, instead of describing it, help them figure it out.

(b) Let $q_2(x) = \frac{x^2}{(1-x)^2}$, $q_3(x) = \frac{x^3}{(1-x)^3}$, etc. Find the 10th-degree Taylor polynomial expansion of $q_2(x)$, $q_3(x)$, and so forth, about $x = 0$. What is going on here??

19. We now have a method for generating the child of a function; describe a method for generating the parent.

20. Find a formula for the number of ways to write a number as the sum of two squares (including zero and negatives).

21. The triangular numbers are 0, 1, 3, 6, 10, 15, ... Can a number be written as the sum of two triangular numbers in more than one way? Find a formula for the number of ways to write a number as the sum of two triangular numbers.

22. Determine the end behavior of $d(n) = c(n)a(n)$. What are the maximum and minimum possible values for $d(n)$, and when do they occur? (As before, please try to think about this problem analytically—it can and was done without a calculator or computer!)

23. If $m = a^2 + b^2 + c^2 + d^2$, write m^2 as the sum of four squares.

Quat you talking bout, Willis?

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Finally an important thing to say here: Saturday is FREE SLURPEE DAY!

10 *The Slurpee Is Free!*

Important Stuff

PROBLEM

Let's take a quick look at the function $b(n)$, which returns the sum of the squares of the reciprocals of the factors of n. For example,

$$b(10) = 1 + \frac{1}{4} + \frac{1}{25} + \frac{1}{100}$$

since the factors of 10 are 1, 2, 5, and 10.

(h) Write out $b(4)$ as the sum of three numbers.

(u) Prove to your tablemates, beyond a reasonable doubt, that $b(4)$ is *less than* the total length of the path from yesterday's handout (the dotted path on today's handout).

(o) Write out $b(6)$ as the sum of four numbers.

(n) Again, prove that $b(6)$ is less than the total length of the dotted path.

(g) Write out $b(12)$ as the sum of six numbers. Is $b(12)$ less than the length of the dotted path?

(j) What about $b(60)$? $b(2520)$?

(e) Say, how long is that dotted path anyway? You were asked to write an expression for this yesterday.

(t) Justify this statement:

For any n, $b(n) < 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots$.

So, now about that dotted path…

Who says a, b, c get to have all the fun in a multi-part problem?
So, this requires your group to go through two standard deviations of thinking about this?

Peg says kids should learn more about 2520, but maybe 2520 wants to maintain its anonymity.

1. Consider the dashed path on today's handout. Which is longer, the dotted path or the dashed path? Both paths start and end in the same place.

We think the "dashed path" should instead be called "Ina's Path."

2. Find the exact length of the dashed path.

3. Prove that the length of the dotted path must be finite.

4. Prove that there is a maximum possible value for $b(n)$.

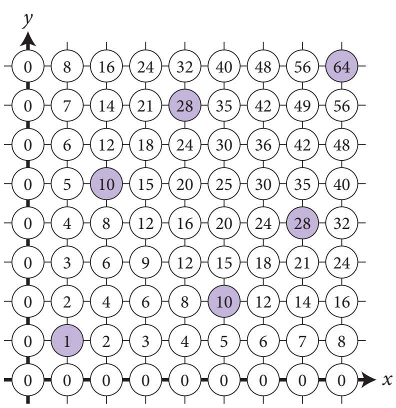
Finally an important thing to say here: Saturday is FREE SLURPEE DAY!

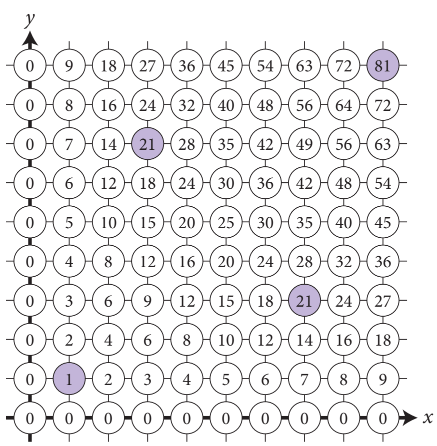
10 *Seriously, The Slurpee Is Free*

More Important Stuff

So, is this stuff really "more important" or just more? We say "more".

Last week, we defined two seemingly different functions. We defined $P(n)$, which counts how many products xy are one more than a multiple of n, when x and y are allowed to range from 0 to $n-1$. Then we defined $\phi(n)$, which counts how many numbers from 1 to n are relatively prime to n. Here are a few examples:





n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$P(n)$	1	1	2	2	4	2	6	4	6	4	10	4	12	6	8	8
$\phi(n)$	1	1	2	2	4	2	6	4	6	4	10	4	12	6	8	8

Golly gee willikers, the two functions give the same values for all n.

Jumpin' Jehosaphat and leapin' lizards, for the love of Pete!

5. (a) Which numbers from 1 to 9 are relatively prime to 9?
(b) In which rows and columns do the products that contribute to $P(9)$ appear?
(c) Which numbers from 1 to 10 are relatively prime to 10?
(d) In which rows and columns do the products that contribute to $P(10)$ appear?

Two numbers are *relatively prime* when they share no common factors greater than 1.

6. (a) Find an n that equals 0 in mod 3, and 1 in mod 10.
(b) Find an n that equals 0 in mod 7, and 1 in mod 10.
(c) Find an n that equals 0 in mod 9, and 1 in mod 10.

Finally an important thing to say here: Saturday is FREE SLURPEE DAY!

10 *No Really, It's Free*

Call today and we'll double your order! You'll get TWO problems in the box for the price of one. And, if you're one of the next 20 to act, we'll throw in a set of CRAZEE BONUS PROBLEMS free! You'll get zeta functions, zeta functions, zeta functions! All you pay is shipping and handling! Money back guarantee. Call now. Call now! Call now! (Seriously, we actually do have an extra problem set for those interested. It's, um, highly algebraic.)

PROBLEM

Last Friday, we observed that the child of the $\phi(n)$ function is the identity function. For example:

$$\phi(1) + \phi(3) + \phi(5) + \phi(15) = 15 \text{ and } \phi(1) + \phi(2) + \phi(7) + \phi(14) = 14$$

Color in the pretty transparencies at your table; they correspond to the multiplication tables used to generate the P function. Stack each group of transparencies on top of each other and discuss what you see. How do the transparencies relate to this observation about the ϕ function?

7. (a) List all the numbers that make up $\phi(1), \phi(3), \phi(5)$, and $\phi(15)$.

(b) Multiply each set of numbers by $\frac{15}{n}$ where n is the input to each $\phi(n)$. For example, multiply all the elements of $\phi(5)$ by $\frac{15}{5} = 3$. What do you get?

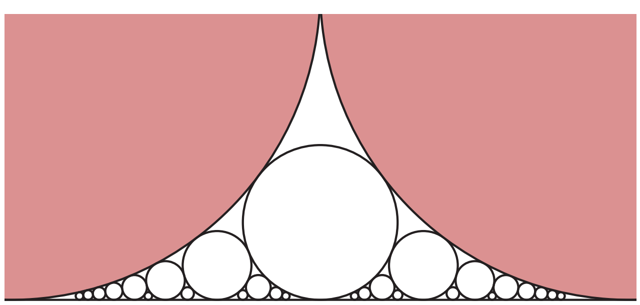
(c) Repeat for 14 by looking at $\phi(1), \phi(2), \phi(7)$, and $\phi(14)$.

(d) Repeat for an interesting number of your choosing.

Neat Stuff

8. Below is a close-up of our favorite diagram, packed with all possible circles! Between each pair of tangent circles, stuff one tangent to them and to the x-axis. Lather, rinse, and you'll end up with an infinite number of circles that are all tangent to each other *and* the x-axis. The shaded regions are the two original circles with diameter 1.

OK, the picture isn't as ridiculous as it could be. If we included an infinite number of circles, it would be. This diagram is blowed up real good!



There is one circle with diameter $\frac{1}{2^2}$ and two circles with diameter $\frac{1}{3^2}$. What other circle diameters will you find, and how many of each size will you find?

Finally an important thing to say here: Saturday is FREE SLURPEE DAY!

No Really, It's Free

9. Investigate a connection between the dotted path, the dashed path, and Pythagorean triples.

10. The centers of *all* the circles on the dotted path are on the same parabola. Find its equation, and sketch an accurate graph of the parabola on top of the circle diagram.

Remember it has to pass through all those centers!

11. Calculus (or a lookup table) can tell you the length of the parabola from Problem 10 from $x = 0$ to $x = 1$. Find a good upper bound for the length of the dotted path.

This isn't to say that Ina's upper bound isn't good, just that a better one can be found by using a parabola instead of using the dashed path.

12. How large can the total area of *all* the circles in Problem 8 get? Is there an upper bound, even though there are an infinite number of circles?

There's a lot of those thingies, but where are they all located?. . .

13. Explore the x-coordinates of the points of tangency of the circles in Problem 8. Specifically, if two neighboring circles are tangent at $x = \frac{a}{b}$ and $x = \frac{c}{d}$, what's the point of tangency of the stuffed-in circle?

14. Show that the left-and-right circles from yesterday's set converge on an interesting point. Your work in Problem 13 may prove very helpful.

15. Yesterday you saw the power series $p(x) = 1 + 2x + 2x^4 + 2x^9 + 2x^{16} + \cdots$.

(a) Use an Nspire or any other method to calculate $(p(x))^4$. Complete this table where $f_4(n)$ is the coefficient of x^n in the polynomial power.

If this problem makes you feel ill, you might be coming down with polynomila.

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$f_4(n)$													

n	13	14	15	16	17	18	19	20	21	22	23	24	25
$f_4(n)$													

(b) Show that f_4 is *not* multiplicative.

(c) Find a function related to f_4 that is multiplicative.

(d) For what do you think $f_4(n)$ might be useful for?

L-functions are named after mathematician Laverne DeFazio. Often collaborating with Feeney, the first L-functions covered the numbers "5, 6, 7, 8" and many were discovered at a brewery in Milwaukee, WI. DeFazio went on to found a successful food company dealing primarily in rabbit stew.

Finally an important thing to say here: Saturday is FREE SLURPEE DAY!

16. Explore the relationship between $c_1(n) = \frac{n}{\phi(n)}$ and $a_1(n) = \frac{n}{\sigma(n)}$. This table may be helpful.

n	$c_1(n)$	$a_1(n)$	$c_1(n)a_1(n)$
1	1	1	1
p	$\dfrac{p}{p-1}$	$\dfrac{p}{p+1}$	
p^2			
p^3			

Tough Stuff

17. As n gets larger and larger, what happens to the value of $c(n)a(n)$? You might prefer to ask the question about the reciprocal, $c_1(n)a_1(n)$, but it's up to you.

Don't be a ζ hata!

18. How many ways are there to write a number as the sum of two squares (including zero and negatives)? There's more than one possible formula!

19. Prove that

$$\sum_{n=1}^{\infty} \frac{\phi(n)}{n^4}$$

is finite.

20. c is a number between 0 and 1. Prove that c is irrational if and only if an *infinite* number of circles from the diagram in Problem 8 intersect the vertical line $x = c$.

21. Multiply the following. What happens? Be specific! And prove it!

$$\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{9}\right)\left(1 - \frac{1}{25}\right)\left(1 - \frac{1}{49}\right)\cdots$$

22. In the diagram from Problem 8, you can make a path goes from $(0,0)$ to $(1,0)$ passing through the centers of *all* the circles, in increasing order by x-coordinate. It zigzags a bit, but the length of the path through any one circle is always known. Is the total length of this path finite, or not? If it's finite, give an upper bound on the length; if it's infinite, prove it.

You can't stop this path! You can only hope to contain it.

Oh, and by the way, the slurpee? It's free. But only on Saturday.

Finally an important thing to say here: Saturday is FREE SLURPEE DAY!

No Really, It's Free

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