Today's problem set is a truly moo-ving experience.

11 *Parent-Child Conference*

Important Stuff

PROBLEM

Last week, we had some practice finding the child of a function. Last Monday we considered the scintillating function $m(n) = 1$ and found its child τ and grandchild u. Today, we're going the other way on m's family tree! Suppose m is the child of a function we'll call z. Then

$$m(n) = z(\text{all divisors of } n) \text{ added together}$$

Let's work it out, starting with $z(1)$. There's only one divisor, so $m(1) = z(1)$, and $z(1) = 1$. What about $z(2)$? Well...

$$\cancel{m(2)}^{\,1} = \cancel{z(1)}^{\,1} + z(2)$$

and $z(2) = 0$. Keep going with $z(3)$ and so on. Fill in the table with the values of z, the parent of m, and then the values of moo, the parent of z.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$moo(n)$	1			0		1				1					
$z(n)$	1	0													
$m(n)$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$\tau(n)$	1	2	2	3	2	4	2	4	3	4	2	6	2	4	4
$u(n)$	1	3	3	6	3	9	3	10	6	9	3	18	3	9	9

n	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
$moo(n)$															
$z(n)$										0					
$m(n)$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$\tau(n)$	5	2	6	2	6	4	4	2	8	3	4	4	6	2	8
$u(n)$	15	3	18	3	18	9	9	3	30	6	9	10	18	3	27

And it came to pass that the sacred cow function moo begat z, which begat m, which begat τ, which begat u.

Careful when calculating $z(3)$, you only use the factors of 3:

$$m(3) = z(1) + z(3)$$

Similarly

$$m(4) = z(1) + z(2) + z(4)$$

Each time, fill in the ones you know... and there should only be one left!

You may feel negatively about some values you're getting for the moo function, but as long as the arithmetic works, it's all good in the Hood. East coast milk brand joke!

Today's problem set is a truly moo-ving experience.

1. Go back to your notes from Week 2 and write down five things that you thought were neat, or things that you're still wondering about.

Dude! We mean business! WRITE FIVE THINGS DOWN. We're counting this as a quiz grade.

2. Find the first ten numerators in this crazee-looking product. What does this have to do with today's problem in the box?

$$\left(\frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\cdots\right)^2=\frac{?}{1^s}+\frac{?}{2^s}+\frac{?}{3^s}+\frac{?}{4^s}+\cdots$$

For example, when you end up seeing a term like $\frac{1}{2^s3^s}$, write that as $\frac{1}{6^s}$. But don't try to simplify something like $\frac{2}{2^s}$ to $\frac{1}{2^{s-1}}$, just leave it so the denominators are all k^s.

3. A function is *multiplicative* if $f(ab) = f(a)\cdot f(b)$ whenever a and b don't have a common factor greater than 1.

(a) If f is multiplicative, explain why $f(5) = f(1)f(5)$ must be true.

(b) Suppose $f(5)$ is nonzero. What does the above equation say about $f(1)$?

(c) If $f(1) > 1$, explain why f *cannot* be multiplicative.

4. Define $s_2(n)$ to be the number of ways to write n as the sum of two squares, where the order and signs of numbers matters. For example, $s_2(10) = 8$ because

$$10 = 3^2+1^2 = (-3)^2+1^2 = 3^2+(-1)^2 = (-3)^2+(-1)^2$$

$$10 = 1^2+3^2 = 1^2+(-3)^2 = (-1)^2+3^2 = (-1)^2+(-3)^2$$

Fill in this table by using today's handout.

The handout should be very helpful. What shape is formed by the eight 10s on this handout? Did you know Pearl Jam's famous "Ten" album is so named because of Moo-kie Blaylock?

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$s_2(n)$	1		4								8		

n	13	14	15	16	17	18	19	20	21	22	23	24	25
$s_2(n)$	8												

5. (a) Determine whether or not s_2 is multiplicative.

(b) Let $S_2(n) = \frac{s_2(n)}{4}$. Does S_2 appear to be multiplicative?

6. Multiply this out:

$$(1+2x+2x^4+2x^9+2x^{16}+2x^{25})^2$$

and write the terms in increasing order of exponent (so you'll write $4x^2$ before $8x^{10}$). Notice anything?

If you say "Yeah, I noticed I already did this problem last week", then keep moo-ving.

7. Define $s_4(n)$ to be the number of ways to write n as the sum of four squares, where the order and signs of numbers

Today's problem set is a truly moo-ving experience.

matters. For example, $s_4(1) = 8$ because

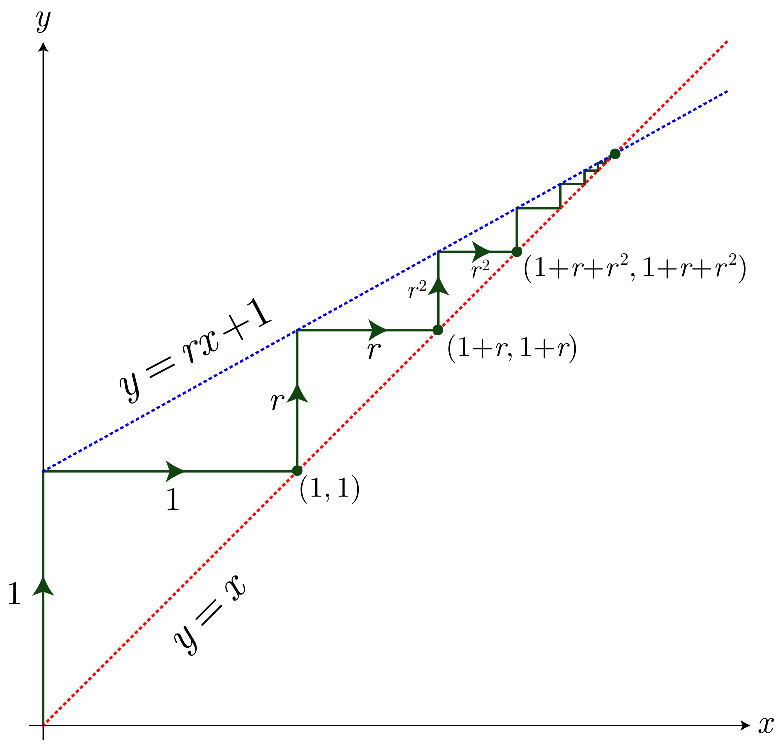
$$\begin{aligned}
1 &= (\pm 1)^2 + 0^2 + 0^2 + 0^2 \\
1 &= 0^2 + (\pm 1)^2 + 0^2 + 0^2 \\
1 &= 0^2 + 0^2 + (\pm 1)^2 + 0^2 \\
1 &= 0^2 + 0^2 + 0^2 + (\pm 1)^2
\end{aligned}$$

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$s_4(n)$	1	8	24	32	24	48	96	64	24	104	144	96	96
n	13	14	15	16	17	18	19	20	21	22	23	24	25
$s_4(n)$	112	192	192	24	144	312	160	144	256	288	192	96	248

(a) Determine whether or not s_4 is multiplicative.
(b) Define a function $S_4(n)$ based on $s_4(n)$ that you think is multiplicative, and test a few examples.

8. Check out this figure:

Yep, that is quite the figure. Feel free to argue about what happens when $r \geq 1$, but it's really a moo-t point.

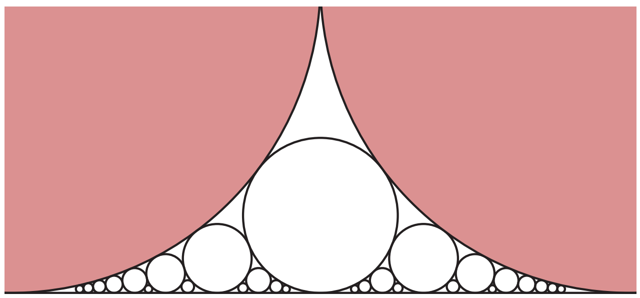


Where do the lines intersect, and what's this got to do with geometric series?

Today's problem set is a truly moo-ving experience.

Circle-y Stuff

9. Below is a close-up of our favorite diagram, packed with all possible circles! Between each pair of tangent circles, stuff one tangent to them and to the x-axis. Lather, rinse, and you'll end up with an infinite number of circles that are all tangent to each other *and* the x-axis. The shaded regions are the two original circles with diameter 1.



There is one circle with diameter $\frac{1}{2^2}$ and two circles with diameter $\frac{1}{3^2}$. What other circle diameters do you find, and how many of each size will you find?

Some of these problems are repeats, some are not. If you're tired of the circles, skip this section. But we'll be sad and moo-dy about it.

10. Investigate the x-coordinates of the centers of the circles (or, if you prefer, their points of tangency with the x-axis), especially when looking at the circles from left to right. Remember, we fixed the two large diameter-1 circles to have centers $(0, \frac{1}{2})$ and $(1, \frac{1}{2})$.

Trivia: Who invaded Spain in the 8th Century? Answer later.

11. (a) Show algebraically that if two tangent circles have diameters $\frac{1}{a^2}$ and $\frac{1}{b^2}$, the next stuff-it-inside circle will have diameter $\frac{1}{(a+b)^2}$.

(b) Show that if a and b are relatively prime, then so are a and $(a + b)$, and so are b and $(a + b)$.

(c) Prove that if two tangent circles in the diagram above have diameters $\frac{1}{a^2}$ and $\frac{1}{b^2}$, then a and b must always be relatively prime.

Fractions help here. If a and b are relatively prime, then the fraction $\frac{a}{b}$ is in lowest terms. So if you moo-tate the fraction $\frac{a+b}{a}$ . . .

12. Look for some Pythagorean triples in right triangles whose hypotenuses are the segments connecting the centers of mutually tangent circles. Can every primitive Pythagorean triple be found in this diagram eventually?

Today's problem set is a truly moo-ving experience.

Neat Stuff

13. Find a rule that works for $moo(n)$ for all the n in the table earlier, especially $n = 30$. For what numbers is $moo(n) = 0$?

Tired of all these puns? The feeling's moo-tual. . .

14. Write out a formula for the sum of an infinite geometric series.

15. Verify each of these formulas with one or two examples.

(a)
$$1 + \frac{1}{x} + \frac{1}{x^2} + \cdots + \frac{1}{x^n} + \cdots = \frac{x}{x-1}$$

(b)
$$1 + \frac{1}{x^2} + \frac{1}{x^4} + \cdots + \frac{1}{x^{2n}} + \cdots = \frac{x^2}{x^2-1}$$

(c)
$$1 + r + r^2 + \cdots + r^n = \frac{1-r^{n+1}}{1-r}$$

16. Let p be any prime. Use today's problem in the box to complete this table.

n	$moo(n)$	$z(n)$	$m(n)$	$\tau(n)$	$u(n)$
1					
p					
p^2					
p^3					
p^4					

What's a cow's favorite Bruce Willis TV series? What's a cow's favorite Cher movie? What's a cow's favorite Michael Jackson dance move? What's a cow's favorite Mets outfielder? What's a cow's favorite Cab Calloway song? What's a cow's favorite Zac Efron movie? What's a cow's favorite Stephenie Meyer book?

17. Multiply this out, again with selective cancellation (all terms should be in the form $\frac{n}{k^s}$). What does this have to do with the today's problem in the box?

$$\left(\frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots\right)^3 = \frac{?}{1^s} + \frac{?}{2^s} + \frac{?}{3^s} + \frac{?}{4^s} + \cdots$$

Notice anything?

It's not alright to shout "Nothing! Absolutely nothing!" here. Save that for Wheel of Fish, a game show where you can earn both moo-lah *and* Moo-nlight Gouramis.

18. Multiply this out:

$$\left(\frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots\right)^0 = \frac{?}{1^s} + \frac{?}{2^s} + \frac{?}{3^s} + \frac{?}{4^s} + \cdots$$

Your mathematical journey is nearing its end...

19. Determine the unique set of coefficients that make the following equation true.

Trivia Answer: the Moo-ps! Well, that's what it says on the card here.

$$\left(\frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\cdots\right)\left(\frac{?}{1^s}+\frac{?}{2^s}+\frac{?}{3^s}+\frac{?}{4^s}+\cdots\right)=1$$

Tough Stuff

20. Repeat Problem 4 for the number of ways to write a number in the form $n = x^2-xy+y^2$ where x and y are integers. It'll probably help if you were here last year.

21. Prove that S_2 and S_4 are multiplicative. Try S_2 first; you may wish to use complex numbers a bit, since the norm of the Gaussian integer $a + bi$ is $a^2 + b^2$. For S_4... um, best of luck.

22. Consider a set of mutually tangent spheres on a plane. Find some relationships between the diameters of the mutually tangent spheres.

Coming soon: Spheres on a Plane! This joke is rated PG, but there is a pretty obvious R-rated version available. It starts with "Enough! I have had it..."

Your mathematical journey is nearing its end...

12 *The Parent Trap*

Important Stuff

PROBLEM

Now let's reach waaay back to Day 1. Remember the σ function? We defined it so that $\sigma(n)$ is the sum of the divisors of n. Let `id` be the parent of σ and `ego` be the grandparent of σ. Fill in this table using yesterday's parent-child connection.

n	1	2	3	4	5	6	7	8	9	10	11	12
`ego`(n)	1					2						
`id`(n)	1											
$\sigma(n)$	1	3	4	7	6	12	8	15	13	18	12	28

Sigma, sure, no problem! What? Yesterday's discussion with those lights might be helpful.

1. Find the first ten numerators in this zany product of infinite sums. What does this have to do with today's first problem in the box?

$$\left(\frac{1}{1^s}+\frac{2}{2^s}+\frac{3}{3^s}+\frac{4}{4^s}+\cdots\right)\left(\frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\cdots\right)$$
$$=\frac{?}{1^s}+\frac{?}{2^s}+\frac{?}{3^s}+\frac{?}{4^s}+\cdots$$

2. Consider this even zanier infinite product of infinite sums:

The only thing better than a zany product is an even zanier product! Anyway. Back to work.

$$\begin{aligned} A = &\left(1+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\cdots\right)\\ &\cdot\left(1+\tfrac{1}{3}+\tfrac{1}{9}+\tfrac{1}{27}+\cdots\right)\\ &\cdot\left(1+\tfrac{1}{5}+\tfrac{1}{25}+\tfrac{1}{5^3}+\cdots\right)\\ &\cdot\left(1+\tfrac{1}{7}+\tfrac{1}{49}+\tfrac{1}{7^3}+\cdots\right)\cdots\\ &\cdot\left(1+\tfrac{1}{p}+\tfrac{1}{p^2}+\tfrac{1}{p^3}+\cdots\right)\cdots \end{aligned}$$

Wow, this goes on forever. Let's think about what this expands *to*, not actually do it. The terms of the expansion come from picking one term from each set of parentheses, then multiplying them together. The final expansion is the sum of *all* such possibilities.

(s) Find a way to get $\frac{1}{12}$ by taking a piece from each factor.

(t) Is this the only way to get $\frac{1}{12}$?

(a) How many ways are there to get $\frac{1}{45}$?

(c) How many ways are there to get $\frac{1}{17}$?

(e) Pick another fraction in the form $\frac{1}{n}$ and describe how to get it in the expansion.

(y) What is the result of the expansion? How big is this product?

The product is big! So big, in fact, that it goes all the way to... We've seen the result of this expansion sometime in Week 1.

3. Consider this ridiculously zany infinite product of infinite sums of wacko numbers:

$$\begin{aligned} B = & \left(1 + \tfrac{1}{4} + \tfrac{1}{16} + \tfrac{1}{4^3} + \cdots\right) \\ & \cdot \left(1 + \tfrac{1}{9} + \tfrac{1}{81} + \tfrac{1}{9^3} + \cdots\right) \\ & \cdot \left(1 + \tfrac{1}{25} + \tfrac{1}{25^2} + \tfrac{1}{25^3} + \cdots\right) \\ & \cdot \left(1 + \tfrac{1}{49} + \tfrac{1}{49^2} + \tfrac{1}{49^3} + \cdots\right) \cdots \\ & \cdot \left(1 + \tfrac{1}{p^2} + \tfrac{1}{p^4} + \tfrac{1}{p^6} + \cdots\right) \cdots \end{aligned}$$

Dang, this goes on forever *too*. Let's ask some incredibly similar questions.

(r) Find a way to get $\frac{1}{144}$ by taking a piece from each factor.

(i) Is this the only way to get $\frac{1}{144}$?

(c) How many ways are there to get $\frac{1}{45^2}$?

(h) How many ways are there to get $\frac{1}{17^2}$?

(a) How many ways are there to get $\frac{1}{20}$? Why?

(r) Pick another fraction in the form $\frac{1}{n^2}$ and describe how to get it in the expansion.

(d) What is the result of the expansion? How big is this infinite product of infinite sums? Infinite, right? Riiiiight?

It's only in this last part that this problem and the last go in separate ways. We've seen the result of this expansion sometime in Week 2.

Your mathematical journey is nearing its end. . .

PROBLEM

Yesterday, we defined s_2, a function counting how many ways you can write numbers as the sum of two squares. We noted that s_2 itself isn't multiplicative since $s_2(1) = 4$, but that $S_2(n) = s_2(n)/4$ seems to be multiplicative. Let R_2 be the parent of S_2. Fill in this table with the values of $R_2(n)$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$R_2(n)$	1		-1						1						
$S_2(n)$	1	1	0	1	2	0	0	1	1	2	0	0	2	0	0

n	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
$R_2(n)$															
$S_2(n)$	1	2	1	0	2	0	0	0	0	3	2	0	0	2	0

Turns out you can find a parent regardless of whether or not the original function is multiplicative. As we'll see though, if the parent is multiplicative, the child is just the same way. Isn't the R_2 function amazing??

4. (a) Write a simple rule that could be used to calculate $R_2(n)$ for any n.

(b) Calculate this:

$$R_2(1) + R_2(2) + R_2(5) + R_2(10) + R_2(13) + R_2(26) + R_2(65) + R_2(130)$$

(c) What is $S_2(130)$? Calculate any way you want it.

Is that the way you need it? Sorry, this joke was written far too late at night.

5. Using what you know about the R_2 function, find and justify a rule for $S_2(p)$ for prime p.

6. Find and justify a rule for $S_2(n)$ for any n. Your work in Problem 4 may help.

7. Suppose that f is a non-zero multiplicative function and g is child. Let $f(3) = a$ and $f(7) = b$.

(a) What is the only possible value for $f(1)$?

(b) Calculate $f(21)$ in terms of a and b.

(c) Write $g(3)$ in terms of a. Remember, g is the child of f.

(d) Write $g(7)$ in terms of b.

(e) Write $g(21)$ in terms of a and b.

(f) Is it true that $g(21) = g(3)g(7)$? For what kind of numbers could this argument be used?

See Problem 3 on Day 11, people.

The parent-child relationship means that $g(3) = f(1) + f(3)$. This will help you find the only solutions to each case.

Neat Stuff

8. Let p be any prime. Complete this table.

n	$\sigma(n)$	$\mu(n)$	$\phi(n)$	$\tau(n)$
1				
p				
p^2				
p^3				
p^4				

OK, that $moo(n)$ function from yesterday? It's really called the Möbius μ function. Please accept this small change with open arms.

9. A *lattice point* is a point with integer coordinates. How many lattice points are on the graph of each of these?
(a) $x^2 + y^2 = 25$
(b) $x^2 + y^2 = 65$
(c) $x^2 + y^2 = 1105$

If you held these graphs above your head, each one would look like a wheel in the sky.

10. Figure out the sequence of missing numerators. Can you do it without performing any algebra?

$$\left(\frac{?}{1^s} + \frac{?}{2^s} + \frac{?}{3^s} + \frac{?}{4^s} + \cdots\right)\left(\frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots\right)$$

$$= \frac{1}{1^s} + \frac{1}{2^s} + \frac{0}{3^s} + \frac{1}{4^s} + \frac{2}{5^s} + \frac{0}{6^s} + \frac{0}{7^s} + \frac{1}{8^s} + \frac{1}{9^s} + \frac{2}{10^s} + \frac{0}{11^s} + \frac{0}{12^s} + \cdots$$

11. Again, figure out the sequence of missing numerators. Holy cow, there's only possible answer here.

$$\left(\frac{?}{1^s} + \frac{?}{2^s} + \frac{?}{3^s} + \frac{?}{4^s} + \cdots\right)\left(\frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots\right) = 1$$

Work faithfully on this problem and a reward awaits!

12. Define $s_4(n)$ to be the number of ways to write n as the sum of four squares, where the order and signs of numbers matters. For example, $s_4(1) = 8$ because

$$\begin{array}{ccccccccc} 1 & = & (\pm 1)^2 & + & 0^2 & + & 0^2 & + & 0^2 \\ 1 & = & 0^2 & + & (\pm 1)^2 & + & 0^2 & + & 0^2 \\ 1 & = & 0^2 & + & 0^2 & + & (\pm 1)^2 & + & 0^2 \\ 1 & = & 0^2 & + & 0^2 & + & 0^2 & + & (\pm 1)^2 \end{array}$$

If you already did this problem on Day 11, look into the future and skip this.

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$s_4(n)$	1	8	24	32	24	48	96	64	24	104	144	96	96

n	13	14	15	16	17	18	19	20	21	22	23	24	25
$s_4(n)$	112	192	192	24	144	312	160	144	256	288	192	96	248

(a) Determine whether or not s_4 is multiplicative.

(b) Define a function S_4 based on s_4 that you think is multiplicative, and test a few examples.

13. Make a conjecture about the value of $S_4(p)$ for prime p.

14. Here's an interesting sequence o' sequences.

Step	Sequence
1	1 1
2	1 2 1
3	1 3 2 3 1
4	1 4 3 2 3 4 1
5	1 5 4 3 5 2 5 3 4 5 1

At step n, look through the last sequence for two consecutive numbers that add to n, and whenever that happens, insert n. Investigate this and look for any interesting connections.

Bowen is loving this problem. Try keeping track of the number of insertions. Say, this might even connect with the circles that were touching each other, when we kept squeezing more into the diagram.

15. As we did with s_2 and s_4, define $s_3(n)$ as how many ways you can write n as the sum of *three* squares (with positions and signs of the three numbers being significant).

(a) Use a power series to help you generate data for s_3 quickly. Or, construct a three-dimensional version of Day 11's handout. Your choice!

(b) Is s_3 multiplicative or can it made multiplicative like we did S_2 and S_4?

(c) Determine r_3, the parent of s_3. See anything cool?

I wonder who's crying now after working on this problem too hard.

16. Use the style of Problem 7 to prove more generally that if g is the child of f and f is multiplicative, then so is g.

Tough Stuff

17. Prove that if g is the child of f and f is *not* multiplicative, then neither is g. Hint: find the smallest n that violates the multiplicativity of f and...

Never mind that Journey, today we take it back to the old school...

18. Categorize all positive integers n that *cannot* be written as the sum of three squares.

19. Prove that any positive integer n can be written as the sum of four squares.

20. Find the exact value of each summation, or show that the sum diverges.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{\sigma(n)}{n^2}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{\mu(n)}{n^2}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{\phi(n)}{n^2}$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{\tau(n)}{n^2}$

21. Define $s_m(n)$ to be the number of ways n can be written as the sum of m squares, where the order and signs of the numbers matters. For which positive integers m is s_m proportional to a multiplicative function?

Don't stop believing that there will be more math problems and bad jokes and references each day until Friday.

Never mind that Journey, today we take it back to the old school...

13 *Parents Just Don't Understand*

Important Stuff

1. Use an Nspire to expand this if you haven't done it yet:

Expand the expression, not THIS.

$$(1 + 2x^1 + 2x^4 + 2x^9 + 2x^{16} + \cdots)^4$$

The coefficient of the x^n term gives the number of ways n can be written as the sum of four squares. We called this function s_4.

But its nickname was Mike.

(l) Determine $s_4(4)$.

(a) Find all the ways to write 4 as the sum of four squares (you don't need to write them all out). Order and signs matter, so $(-1)^2 + (-1)^2 + 1^2 + 1^2$ is different from $1^2 + (-1)^2 + 1^2 + (-1)^2$.

(na) Is s_4 multiplicative? Explain.

PROBLEM

Let $S_4(n) = \frac{s_4(n)}{8}$ and let R_4 be the parent of S_4. Fill in this table with the values of $R_4(n)$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$R_4(n)$	1			0	5										
$S_4(n)$	1	3	4	3	6	12	8	3	13	18	12	12	14	24	24

n	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
$R_4(n)$															
$S_4(n)$	3	18	39	20	18	32	36	24	12	31	42	40	24	30	72

2. Compute the following product. Keep going until you notice something amazing!!

It takes Problem 2 to make a thing go right.

$$\left(\frac{1}{1^s} + \frac{2}{2^s} + \frac{3}{3^s} + \frac{0}{4^s} + \frac{5}{5^s} + \frac{6}{6^s} + \frac{7}{7^s} + \frac{0}{8^s} + \frac{9}{9^s} + \cdots\right)\left(\frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots\right) = \text{hmmmm}$$

Never mind that Journey, today we take it back to the old school...

Parents Just Don't Understand

3. Look back at the last few days' expansion problems. Describe what happens when you multiply through by

$$\left(\frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots\right)$$

What form must the other expression have for this to work?

4. Check out this table of values of the Möbius μ function.

Ch-ch-ch-check it out! Wha-what's it all about? Wor-wor-work it out! Let's turn this... never mind. This is the function we called "moo" on Monday.

n	factorization of n	$\mu(n)$	n	factorization of n	$\mu(n)$
1	1	1	16	2^4	0
2	2	-1	18	$2 \cdot 3^2$	0
3	3	-1	20	$2^2 \cdot 5$	0
4	2^2	0	21	$3 \cdot 7$	1
5	5	-1	24	$2^3 \cdot 3$	0
6	$2 \cdot 3$	1	25	5^2	0
7	7	-1	30	$2 \cdot 3 \cdot 5$	-1
8	2^3	0	35	$5 \cdot 7$	1
9	3^2	0	36	$2^2 \cdot 3^2$	0
10	$2 \cdot 5$	1	60	$2^2 \cdot 3 \cdot 5$	0
11	11	-1	77	$7 \cdot 11$	1
12	$2^2 \cdot 3$	0	99	$3^2 \cdot 11$	0
14	$2 \cdot 7$	1	210	$2 \cdot 3 \cdot 5 \cdot 7$	1
15	$3 \cdot 5$	1	2310	$2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$	-1

Use the table to write a rule to calculate $\mu(n)$ for any n. Use your rule to calculate $\mu(120)$, $\mu(5005)$ and $\mu(30030)$.

Your rule for μ can be a sentence or two, it doesn't have to contain complicated symbols.

5. Look at the problem in the box from Day 11, then figure out the sequence of missing numerators in this equation below. Try to do it without performing any algebra.

That right side is better known as "1". What again did you say happens when you multiply through by $\left(\frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots\right)$?

$$\left(\frac{?}{1^s} + \frac{?}{2^s} + \frac{?}{3^s} + \frac{?}{4^s} + \frac{?}{5^s} + \frac{?}{6^s} + \cdots\right)\left(\frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \cdots\right)$$

$$= \frac{1}{1^s} + \frac{0}{2^s} + \frac{0}{3^s} + \frac{0}{4^s} + \cdots$$

6. Find the first six numerators in this insane product of infinite sums. What does this have to do with today's problem in the box?

It's insane, got no brain!

$$\left(\frac{1}{1^s} + \frac{3}{2^s} + \frac{4}{3^s} + \frac{3}{4^s} + \frac{6}{5^s} + \frac{12}{6^s} + \cdots\right)\left(\frac{1}{1^s} + \frac{-1}{2^s} + \frac{-1}{3^s} + \frac{0}{4^s} + \frac{-1}{5^s} + \frac{1}{6^s} + \cdots\right)$$

$$= \frac{?}{1^s} + \frac{?}{2^s} + \frac{?}{3^s} + \frac{?}{4^s} + \frac{?}{5^s} + \frac{?}{6^s} + \cdots$$

Never mind that Journey, today we take it back to the old school...

7. Patty believes that

$$1 + \tfrac{1}{2^2} + \tfrac{1}{3^2} + \tfrac{1}{4^2} + \cdots = \left(1 + \tfrac{1}{4} + \tfrac{1}{16} + \tfrac{1}{4^3} + \cdots\right) \cdot \left(1 + \tfrac{1}{9} + \tfrac{1}{81} + \tfrac{1}{9^3} + \cdots\right) \cdot \left(1 + \tfrac{1}{25} + \tfrac{1}{25^2} + \tfrac{1}{25^3} + \cdots\right) \cdot \left(1 + \tfrac{1}{49} + \tfrac{1}{49^2} + \tfrac{1}{49^3} + \cdots\right) \cdots \cdot \left(1 + \tfrac{1}{p^2} + \tfrac{1}{p^4} + \tfrac{1}{p^6} + \cdots\right) \cdots$$

Is she right? Why or why not?

This is another Overlong Product of Powers. The question is, though: are you down with OPP?

8. Here we go with another scenario! At least this time the sums are finite.

Overlong products of primes! Yeah, you know me!

$$M = \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{5}\right)\left(1 - \frac{1}{7}\right) \cdots \left(1 - \frac{1}{p}\right) \cdots$$

Wouldn't you know it, these just keep going on forever. Some more questions!!

(m) Will you find a $\frac{1}{15}$ term in this product? Why or why not? If so, whatzits sign?

(e) Will you find a $\frac{1}{18}$ term in this product? Why or why not? If so, whatzits sign?

(g) What's the sign of $\frac{1}{17}$?

(h) What happens with $\frac{1}{20}$? $\frac{1}{30}$?

(a) What denominators do you get, and with what signs?

(n) What is the result of the expansion? Holy cow!

Neat Stuff

9. So now let's look at $\frac{1}{M}$:

Take each term from the original product and push it into the denominator. Add salt and pepper as needed.

$$\frac{1}{M} = \left(\frac{1}{1 - \frac{1}{2}}\right)\left(\frac{1}{1 - \frac{1}{3}}\right)\left(\frac{1}{1 - \frac{1}{5}}\right)\left(\frac{1}{1 - \frac{1}{7}}\right) \cdots$$

Hey... wait a minute... these are all in the form $\frac{1}{1-r}$, like a geometric series! Whoomp!

There it is!

(j) Unravel each geometric series into its terms. For example, $\frac{1}{1-\frac{1}{2}} = 1 + \frac{1}{2} + \frac{1}{4} + \cdots$.

(o) Now multiply out the new $\frac{1}{M}$, if you haven't already.

(e) How big is $\frac{1}{M}$? What does that say about the value of M?!

Never mind that Journey, today we take it back to the old school...

10. Repeat the last two problems with this infinite product:

$$N = \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{5^2}\right)\left(1 - \frac{1}{7^2}\right)\cdots\left(1 - \frac{1}{p^2}\right)\cdots$$

What does the expansion of N look like? Find the value of N as exactly as you can by exploring the geometric series buried inside $\frac{1}{N}$. There has to be a value, since N must be between 0 and 1!

This is pretty tough, but if you aren't sure what comes next, bust a move back to the steps you followed in Problems 8 and 9.

Between 0 and 1 factorial, eh? It's like that, and that's the way it is.

11. (a) If two positive integers are picked at random, what is the probability that they are *both* multiples of 2?

(b) What is the probability that at least one of the two numbers *isn't* a multiple of 2? Psst: use $1 - p$.

(c) What is the probability that at least one of the two numbers *isn't* a multiple of 3?

(d) What is the probability that the two numbers don't have a common factor of 5? (This is the same as the last question.)

(e) 7? 11? Slurpee?

(f) What is the probability that two positive integers picked at random won't have a common factor of 2, 3, or 5?

(g) Write an expression for the probability that two positive integers picked at random will share *no* common factors.

This is the worst problem phrasing ever constructed, but that's how we roll. The choice is yours. Hey, we can't help it that 7 and 11 are consecutive primes.

12. What infinite series in the style of Problem 3 can be multiplied onto something to obtain its grandchild?

13. What infinite series in the style of Problem 3 can be multiplied onto something to obtain its grandparent?

Seth wonders why there are grandchildren and grandparents, but no grandmasters...

14. Let p be any prime. Complete this table.

"gp" is not a reference to Gangster's Paradise.

n	ggp	gp	parent of m	$m(n) = 1$	child of m	gc	ggc
1							
p							
p^2							
p^3							
p^4							

Never mind that Journey, today we take it back to the old school...

15. Let p be any prime. Complete this table.

n	ggp	gp	parent of `id`	$\texttt{id}(n) = n$	child of `id`	gc	ggc
1							
p							
p^2							
p^3							
p^4							

16. Show that for any integer $n \geq 2$,

$$\phi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right).$$

What do those symbols mean anyway?

17. Investigate the behavior of this function.

$$f(n) = \frac{\sum_{k=1}^{n} \phi(k)}{n^2}$$

If you got to this problem, I got to say it was a good day.

Tough Stuff

18. What is $\lim_{n\to\infty} \frac{1}{n^2} \sum_{k=1}^{n} s_4(k)$? Prove it.

19. Read the 14 proofs at this website:
`http://www.secamlocal.ex.ac.uk/people/staff/rjchapma/etc/zeta2.pdf`
Which one do you like best? We hope at least one of them amazes you!

We like the proofs, the proofs that go boom. We're Sara and Maura and we like the boom.

20. Suppose you have an unlimited supply of beads with k different colors. How many distinct necklaces with length n can you make? Try to find a way to solve this problem using Möbius inversion.

If R and B are two colors (raw umber and burnt umber), RBRRR and RRBRR are not distinct necklaces since they are related by a circular shift.

21. Find a nice rule for $s_3(n)$, the number of ways to write n as the sum of three squares.

22. For what m is s_m proportional to a multiplicative function?

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Today we take it back to My Old School, with everything older than (anyone on) the hills.

14 *SS2P 2009: Rated PG*

Important Stuff

SS2P: Strictly Sums To Products!

PROBLEM

The s_2 function is a bit of a mess. It gets up, gets down, it's zero a lot, and when it's not zero it's almost always a multiple of 4. One way to deal with a bizarre function like this one is to turn it into a running average.

So, any way you like, compute the average value of $s_2(n)$ when

(t) n goes from 1 to 25

(e) n goes from 1 to 49

(r) n goes from 1 to 75

(i) n goes from 1 to 108

You may find the handout from Day helpful, mostly.

This function would lose its mind in Detroit Rock City.

To everything, turn, turn, turn it into a running average.

Dang, this number seems to have disappeared faster than the guy who did "Spirit in the Sky".

1. Sandy believes that

$$\begin{aligned} 1+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\cdots = &\left(1+\frac{1}{2^s}+\frac{1}{4^s}+\frac{1}{8^s}+\cdots\right)\\ &\cdot\left(1+\frac{1}{3^s}+\frac{1}{9^s}+\frac{1}{27^s}+\cdots\right)\\ &\cdot\left(1+\frac{1}{5^s}+\frac{1}{25^s}+\frac{1}{5^{3s}}+\cdots\right)\\ &\cdot\left(1+\frac{1}{7^s}+\frac{1}{7^{2s}}+\frac{1}{7^{3s}}+\cdots\right)\cdots\\ &\cdot\left(1+\frac{1}{p^s}+\frac{1}{p^{2s}}+\frac{1}{p^{3s}}+\cdots\right)\cdots \end{aligned}$$

Is she right? Why or why not?

Sandy is all right now. They're bad references, but at least they're free.

2. Each of the infinite sums above, like

$$1+\frac{1}{2^s}+\frac{1}{4^s}+\frac{1}{8^s}+\cdots,$$

is a geometric series. Find the value of each geometric series. Rewrite the messy equation above as simply as you can.

3. **Warning:** Notation ahead!

$$\sum_{n=1}^{5} f(n) = f(1) + f(2) + f(3) + f(4) + f(5)$$

$$\sum_{n=1}^{\infty} f(n) = f(1) + f(2) + f(3) + f(4) + f(5) + \cdots$$

$$\prod_{n=1}^{5} f(n) = f(1) \cdot f(2) \cdot f(3) \cdot f(4) \cdot f(5)$$

$$\prod_{p} f(p) = f(2) \cdot f(3) \cdot f(5) \cdot f(7) \cdot f(11) \cdots$$

Write the messy equation from the last problem using as-clean-as-you-can notation, then celebrate by marching in a parade.

A parade? What? Oh, right, we had one of those. But today it's a classic rock Hit Parade!

4. Let $s = 1$ in the messy equation. What happens? Use this to prove that there must be infinitely many prime numbers.

5. Calculate enough terms of this infinite product so that you can identify what the answer is.

$$\left(1 - \frac{1}{2^s}\right)\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{5^s}\right)\left(1 - \frac{1}{7^s}\right)\cdots\left(1 - \frac{1}{p^s}\right)\cdots$$

$$= \frac{?}{1^s} + \frac{?}{2^s} + \frac{?}{3^s} + \frac{?}{4^s} + \frac{?}{5^s} + \frac{?}{6^s} + \cdots$$

Be strategic about expansion; what kinds of terms will you get?

Curses! FOIL'd again! I would've gotten away with it too, if it weren't for those meddling kids.

6. Multiply this out. What happens?

$$\left(\frac{1}{1^s} + \frac{-1}{2^s} + \frac{-1}{3^s} + \frac{0}{4^s} + \frac{-1}{5^s} + \frac{1}{6^s} + \cdots + \frac{\mu(n)}{n^2} + \cdots\right) \cdot$$

$$\left(\frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \cdots + \frac{1}{n^2} + \cdots\right) = \text{hmmmm}$$

7. Find the result of this product. Use the last two problems, buddy/buddette!

$$\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{9}\right)\left(1 - \frac{1}{25}\right)\left(1 - \frac{1}{49}\right)\cdots\left(1 - \frac{1}{p^2}\right)\cdots$$

Review Your Stuff

The final day of this course is mostly taken up by review problems. So we think it would be a good idea for groups to form some summarizing questions that come out of whatever you might find valuable in this course. So, we want your table to write two problems on any subject that has cropped up in the course.

Here are some topics you might consider writing problems about.

Multiplicative functions	σ, τ, ϕ, μ
Modular arithmetic	Parents and children
Circles and summations	Power series
Convergence and divergence	Old school rap

The goal is to create a review whose problems get at the fact that we've come a long, long way in three weeks. The problems should help others synthesize their learning of the aforementioned topics.

So, don't write any stumpers; consider yourself writing two problems that could both fit into "Important Stuff." If your table wants to write more than two, thats fine, and the extra questions can be a little more "Neat" or "Tough." We reserve the right to combine, edit, change, ignore, or otherwise mangle your problems. And it's ok to be funny, as long as it doesnt get in the way of the math.

Neat Stuff

8. What does this equal? Give a justification.

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} s_2(k)}{n}$$

9. Imagine two dice with an infinite number of sides, labeled 1 to . . . um, yeah. Two of these presumably spherical dice are rolled, and a result is calculated: the greatest common divisor of the two numbers rolled.

(a) Try this a few times. Randomly pick ten pairs of five- or six-digit numbers, then calculate the greatest common divisor of each pair. Anything surprising?

(b) Explain why it's exactly 4 times more likely for the result to be 1 than for it to be 2.

(c) How many times more likely is it for the result to be 1 than 3?

(d) Find the exact probability that the result is 1.

10. Use a result from this week to prove that no positive integer can have more factors that are "3 mod 4" than factors that are "1 mod 4".

11. (a) What's the formula for the area of a circle?

(b) What's the formula for the volume of a sphere?

(c) What's the formula for the, uh, hypervolume of a four-dimensional, uh, hypersphere?

Either use calculus or look this up. Easy choice, right?? We hope the answer is still surprising.

12. In today's box you calculated the long-term average value of s_2. Try again with s_4, and see if you find anything interesting.

13. Here's a grid of n^2 fractions:

$$\begin{array}{ccccc} \frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} \\ \frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \cdots & \frac{2}{n} \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{n}{1} & \frac{n}{2} & \frac{n}{3} & \cdots & \frac{n}{n} \end{array}$$

As n grows, what proportion of the fractions are in lowest terms? $\frac{6}{5}$ is in lowest terms, but $\frac{6}{4}$ isn't.

14. Show that

$$\sum_{d|n} |\mu(d)| = 2^{\text{number of distinct primes dividing } n}$$

15. You are the first contestant on the "Showcase Showdown" of Price is Right, and on your first spin you get 65 cents. Are you more likely to win by spinning again and risking going over \$1.00, or by staying on 65 cents? Rigorously defend your logic.

Tough Stuff

16. What does this infinite product equal?

$$\left(1+\frac{1}{4}\right)\left(1+\frac{1}{9}\right)\left(1+\frac{1}{25}\right)\left(1+\frac{1}{49}\right)\cdots\left(1+\frac{1}{p^2}\right)\cdots$$

17. Prove that no positive integer can have more factors that are "2 mod 3" than factors that are "1 mod 3". Generalize to other mods. . . if possible.

18. Find a way to generate all of the Pythagorean triples in which *the two leg lengths* are one away from each other. One example is 21, 20, 29.

19. Solve the continuous version of the "Showcase Showdown" problem above, where numbers are picked continuously from 0 to 1 instead of discretely by increments of 0.05. Find the cutoff number n where it's correct to stay when you get more than n on the first try, and to go again with less than n.

Today we take it back to My Old School, with everything older than (anyone on) the hills.

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Thanks everyone! Enjoy the rest of your summer.

15 *Last Call*

Supremely Unimportant Stuff

0. Let $f(n)$ be the height of the day number on Day n's problem set. Is f linear, quadratic, exponential, or something else? If $f(1) = 1$ unit, is f multiplicative?

Ask Darryl why the 15 isn't large enough to take up an entire page.

Your Stuff

4. Notice that

$$\sigma(8) = \sigma(7) + \sigma(6) - \sigma(3) - \sigma(1)$$

Is it also true that $\sigma(9) = \sigma(8) + \sigma(7) - \sigma(4) - \sigma(2)$? How long does this last? Can you "fix it" when it breaks?

Hm, spoken like someone from the Number Theory working group, so a special note for them: I hear $\mathbb{Z}[\sqrt{-163}]$ has unique prime factorization! What's up with that?

8. Find the solutions to
(a) $x^3 = 1 \bmod 7$
(b) $x^3 = 1 \bmod 13$
(c) $x^3 = 1 \bmod 19$
What do you notice in each case?

Hm, you could use thes to solve $x^3 = 1 \bmod 1729$, which is a very interesting cab number!

10. Find all solutions to $x^2 = 4$ in mod 145 using the factorization $145 = 5 \cdot 29$.

0. Find all solutions to $x^3 - x = 4$ in mod 170.

12. Why does the array model work for $\phi(21)$ but doesn't work for $\phi(20)$?

$$\phi(21) = (3-1)\cdot(7-1) \qquad \phi(20) \neq (4-1)\cdot(5-1)$$

	1	6
1	1	6
2	2	12

↑ $\phi(21)$

	1	4
1	1	4
3	3	12

↑ $\neq \phi(20)$

Thanks everyone! Enjoy the rest of your summer.

9. Complete this table giving numbers that have one answer in mod 12, and a second answer in mod 7.

6							6					
5	12					5						
4					4							11
3				3							10	
2			2							9		
1		1							8			
0	0							7				
	0	1	2	3	4	5	6	7	8	9	10	11

(vertical axis: number mod 7; horizontal axis: number mod 12)

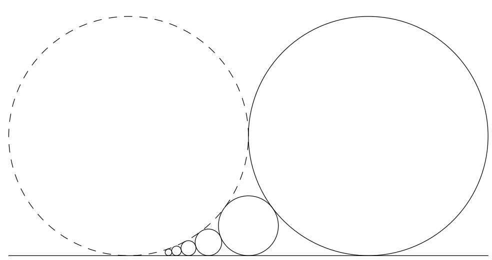
What number between 0 and 84 is 1 in mod 12 and 0 in mod 7? What number is 0 in mod 12 and 1 in mod 7?

9. Repeat the above problem for mod 8 and mod 12. What happens? Is there a number that is 1 in mod 8 and 0 in mod 12?

Commentary from Table 9: "Art is funny." Is he a clown? Does he amuse you? (Yes.)

4. Consider the infinite sequence of circles from Day 8's handout. Find the total circumference of the circles (ignore the dotted circle, whose name is Art).

As a reminder, the sum of the diameters of these circles ended up being this crazy number, $\frac{\pi^2}{6}$.



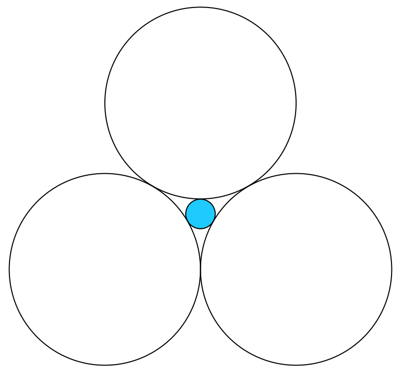
4. Consider the sequence of circles from Day 9, the ones that go left-right-left-right. For each circle, find the x-coordinate of its center. As more and more circles are constructed, what happens to the centers? Specifically, do these circles' centers head toward the point $(\frac{6}{\pi^2}, 0)$?

The circles go left, right, left, right, like an N*SYNC dance number. Bye bye bye PCMI!

Thanks everyone! Enjoy the rest of your summer.

12. The diagram below represents Felipe, his friend Jennifer and his other friend Jennifer. All three of them are shown as circles with diameter 1 that are tangent to each other. Now plop a circle in the middle and call him Adnan (the shaded circle below)!

And lo, a PLOP was heard through the land, and they called him Adnan.



Descartes found this (totally bodacious) formula relating the four diameters; as before, each variable is the *reciprocal* of the real diameter:

$$(a+b+c+d)^2 = 2(a^2+b^2+c^2+d^2)$$

Amazing how simple this formula is, and how similar it is to the case of using a tangent line. Hey wait, what happens when you let $d = 0$? What's up with that?

(a) What is Adnan's diameter?

(b) Now do it again, plopping yet another circle called Marty, between Adnan and the Jennifers. What is Marty's diameter?

Marty would also like to hang out with Felipe, but can't reach across Adnan.

(c) As more circles are plopped, what happens to their diameters? Do the diameters form a recognizable sequence?

2. We've been computing the ways to write any integer as the sum of two squares. As n increases, does the number of ways generally increase? Is there a way to predict the density of the numbers that can be written as the sum of two squares?

You could define a new function that gives 1 when the number *can* be written as the sum of two squares, and 0 when it can't. There's a nice way to do this with power series, but it's not easy to find.

2. In the box from Day 14, as n increases, what happens to the average value of $s_2(n)$? How is this *physically* and/or geometrically related to the handout from Day 11, which gave the $x^2 + y^2$ value at each location (x, y)?

Thanks everyone! Enjoy the rest of your summer.

3. Fill in the following table.

$\zeta(s)^n$	Calculation	Result
$\zeta(s)^{-1}$		
$\zeta(s)^{0}$		
$\zeta(s)^{1}$	$\frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\cdots$	$\frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\cdots$
$\zeta(s)^{2}$	$\left(\frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\cdots\right)^2$	$\frac{\ }{1^s}+\frac{\ }{2^s}+\frac{\ }{3^s}+\frac{\ }{4^s}+\cdots$
$\zeta(s)^{3}$		

Make a connection between what you see here and one of the "problems in the box".

That crazy squiggle is a "zeta", not a "weird C/S lookin' thing". Others may call $\zeta(s)$ the "Riemann zeta function" but you say it's just a friend. What does multiplying by $\zeta(s)$ seem to do to a function? Dividing?

3. Prove that if f is multiplicative, then its child g must also be multiplicative.

If Chucky and his bride have a child, can the child reproduce? [Seriously, a "Bride of Chucky" gag? It wasn't us.]

10. Given a function, how do you obtain the child? parent? grandchild? grandparent? Take a function and calculate its grandparent and grandchild functions, then multiply those together after writing them as $\frac{?}{1^s}+\frac{?}{2^s}+\cdots$. What happens?! Wow!

7. You know that a multiplicative function satisfies $f(xy) = f(x)f(y)$. Or we hope you do! Find a function for which each of these is true for any choices of the variables.

(k) $\texttt{traci}(x+y) = \texttt{traci}(x) + \texttt{traci}(y)$
(e) $\texttt{randy}(x+y) = \texttt{randy}(x) \cdot \texttt{randy}(y)$
(n) $\texttt{aaron}(xy) = \texttt{aaron}(x) + \texttt{aaron}(y)$
(t) $\texttt{chris}(x+y) = (\texttt{chris}(x))^y$

We apologize for the bizarre function names, which serve no purpose other than for us to be Equal Opportunity Name-Droppers.

7. Given any integer $n > 2$, describe how to find a sequence of n consecutive *non*-primes. Does this contradict the earlier finding that there are an infinite number of primes?

12. Is there a "family tree" of all the functions we've studied these three weeks? If so, what does it look like? It may help to think that functions in the form $f(n) = x^n$ are all "related" though this isn't a parent-child relationship.

Maybe they're married or in the same high school class or something, we're really stretching the analogy for no good reason here.

Thanks everyone! Enjoy the rest of your summer.

1. Show that $x^2 + y^2 = 2003$ has no integer solutions.

(k) If you had easy access to complete tables for all functions we have studied, which would you go to to show this result immediately?

(a) What are the possible values of x^2 in mod 4?

(t) Based on this, what can you say about the possible values of $x^2 + y^2$ in mod 4?

(h) Can 2009 be written as the sum of two squares? If so, find one way to do it.

(y) When is the next year that *cannot* be written as the sum of two squares? What will be the top song that year?

Clearly it will be the re-release of "Cars That Go Boom".

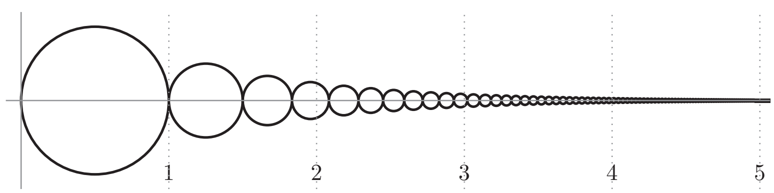
8. How many lattice points are on the graph of $x^2 + y^2 = 3530$? What are they?

Eww; "What are they?" is much tougher than "How many" for this kind of problem.

Our Stuff

1. Here are some circles. Hey, they sort of look like a torus line in a parade, and maybe those two leading the line are Amelia and Carol. Each circle is constructed so that the nth circle has diameter is $\frac{1}{n}$. Find the total circumference and area of *all* the circles built this way.

A Torus Line: great math song, or greatest math song? You decide.



2. Consider m and n, relatively prime integers. Suppose a is 0 in mod m and 1 in mod n, and b is 1 in mod m and 0 in mod n. If $0 \leq a, b \leq mn$, show that $b = mn - a + 1$.

3. Define a function λ that gives $\lambda(n) = 1$ if n has an even number of prime factors and $\lambda(n) = -1$ if n has an odd number of prime factors. This function takes into account the number of times that a prime factor is repeated. So, for example, $\lambda(1) = 1$ because 1 has zero prime factors, $\lambda(2) = -1$ because 2 has one prime factor (namely, 2). The value of $\lambda(4) = 1$ because $4 = 2^2$ so it has the prime factor 2 appearing twice. Fill in this table with the values of λ, its child `louis` and its parent `mary`.

Now we've gone from cows to sheep. The moo function and this lamb-duh function seem pretty similar.

Thanks everyone! Enjoy the rest of your summer.

Last Call

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
mary(n)	1														
$\lambda(n)$	1	-1	-1	1											
louis(n)	1														

n	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
mary(n)	1														
$\lambda(n)$	1														
louis(n)	1														

4. Prove that if a, b, c are positive integers and

$$(a+b+c+d)^2 = 2(a^2+b^2+c^2+d^2)$$

then d must also be an integer. What does this imply about these circle packing diagrams?

Warning: this statement might not be true! And that last statement definitely isn't true. But the previous two statements are both lies.

Tough Stuff

5. Define $\texttt{lori}(n) = \sum_{k=1}^{n} \lambda(k)$. Is $\texttt{lori}(n) \le 0$ for all $n > 1$? Prove or find a counter-example.

6. Two days ago we showed that the equation

$$x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\cdots = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{7!} + \cdots$$

leads to a proof that $1 + \frac{1}{4} + \frac{1}{9} + \cdots = \frac{\pi^2}{6}$. What about the x^5 term? Is there some formula involving 120 in there?

7. Calculate $\displaystyle\sum_{n=2}^{\infty}\sum_{m=1}^{n-1} \frac{1}{(mn)^2}$.

8. Calculate $\displaystyle\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{(mn)^2}$. It'll be related to $\frac{\pi^2}{6}$.

9. Use the previous two facts and a lovely grid to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

10. Use a similar setup to find the exact value of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^6}$.

11. Explain why $e^{\pi\sqrt{163}}$ is an integer. Amazing!