2 *Repeating Matters*

Important Stuff

We're going to start with doing the same thing, over and over. The *Mabbott sequence* is one of the least famous known in math. It starts with 2, then 2 again, then each new term is *twice* the one before it, plus *three times* the one before that. A slightly more formal definition is

$$M(0) = 2$$
$$M(1) = 2$$
$$M(n) = 2 \cdot M(n-1) + 3 \cdot M(n-2) \quad \text{if } n > 1$$

For example, $M(2) = 2 \cdot M(1) + 3 \cdot M(0)$, then $M(3) = 2 \cdot M(2) + 3 \cdot M(1)$, then... Use the values of $M(0)$ and $M(1)$ to find $M(2)$, then use the values... then lather, rinse, and...

PROBLEM

(a) For the Mabbott sequence, determine $M(0)$ through $M(8)$.

(n) Use patterns in $M(n)$ to find a way to calculate $M(12)$ directly without calculating $M(10)$ and $M(11)$.

(d) Your table will be given four new pairs of starting numbers. For each pair, determine the first nine numbers (including the two givens). Notice anything?

(y) Describe some similarities between the five sequences your table worked with.

1. Find a solution to this system of equations:

$$A + B = 2$$
$$3A - B = 2$$

2. Here's a recursive definition for the sequence $0, 1, 3, 6, 10 \ldots$:

"Recursive" doesn't mean "cursive again."

$$s(0) = 0, \qquad s(n) = s(n-1) + n \quad \text{if } n > 0$$

(a) Determine $s(9)$.
(b) Determine $s(100)$.

3. Write a recursive definition for $c(n)$ that fits the sequence $3, 6, 9, 12, 15, \ldots$

This means $c(0)$ should be 3, $c(1)$ should be 6, and $c(221)$ should be 666. Uh oh.

4. Write a recursive definition for $d(n)$ that fits the sequence $3, 6, 12, 24, 48, \ldots$

5. Determine the sum $c(0) + c(1) + c(2) + \cdots + c(6)$ as simply as you can, without a calculator.

6. Determine the sum $d(0)+d(1)+d(2)+\cdots+d(6)$ as simply as you can, ideally without a calculator.

7. Calculate each of these.
(a) $(5+\sqrt{2})+(5-\sqrt{2})$
(b) $(5+\sqrt{2})\cdot(5-\sqrt{2})$

8. Find two numbers with the given sum s and product p.

(a) $s=10, p=25$
(b) $s=10, p=24$
(c) $s=10, p=23$
(d) $s=10, p=22$
(e) $s=10, p=21$
(f) $s=10, p=20$
(g) $s=10, p=1$
(h) $s=10, p=-1$
(i) $s=10, p=-299$
(j) $s=100, p=2451$

Neat Stuff

9. A "Mabbott-like" sequence is defined by

$$R(0) = 5$$
$$R(1) = 19$$
$$R(n) = 2R(n-1)+3R(n-2) \quad \text{if } n>1$$

Find a closed rule, such as $R(n)=3^n$, that matches the sequence, then used the closed rule to find $R(10)$.

10. Find a solution to this system of equations:

$$A+B = 5$$
$$3A-B = 19$$

11. A sequence is defined by

$$v(0) = 7$$
$$v(1) = 23$$
$$v(n) = -v(n-1)-v(n-2) \quad \text{if } n>1$$

Determine $v(227)$.

Jackée would be proud.

12. Without a calculator, estimate the number of digits in $F(1000)$, a really big Fibonacci number.

13. Which Fibonacci numbers are even, and which are odd? Explain why this happens.

14. Which Fibonacci numbers are multiples of 5? Explain why this happens.

Fibonacci numbers are like marching soldiers in The Wizard of Oz. Oh, ee, oh. . . oh, ee, oh. . .

15. The *Lucas sequence* is like the Fibonacci sequence, except it starts with 2 and 1 instead of 0 and 1:

$$L(0) = 2$$
$$L(1) = 1$$
$$L(n) = L(n-1) + L(n-2) \quad \text{if } n > 1$$

$L(2) = 3, L(3) = 4, L(4) = 7$. The Lucas sequence lives on the second floor. Please don't read literature about Fibonacci and Lucas, so that you can find and prove results on your own.

Find as many relationships as you can between the numbers in the Lucas sequence and the numbers in the Fibonacci sequence. Try to prove them!

16. The *Quagmire sequence* is the sum of the Lucas and Fibonacci sequences:

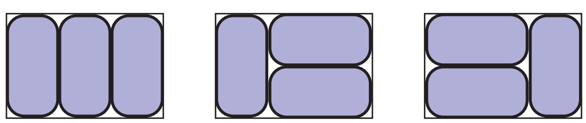
$$Q(n) = L(n) + F(n)$$

Figure out what you can about the Quagmire sequence, and any new relationships you can figure out between the Lucas and Fibonacci sequences.

Giggity giggity.

17. Write a recursive rule for $h(n)$ that fits the sequence $1, 10, 44, 160, 536, 1720, 5384 \ldots$

18. In terms of n, how many ways are there to tile a 2-by-n rectangle with identical 1-by-2 dominoes? Consider any rotations or reflections to be *different* tilings: there are 3 tilings for the 2-by-3 rectangle. Why look, here they are!!



19. In terms of n, how many binary sequences of length n do not have consecutive zeros?

A *binary sequence* is made up of all ones and zeros. For $n = 2$ there are four binary sequences: 00, 01, 10, and 11.

20. Without a calculator, determine the units (ones) digit of $F(100)$ and of $F(1000)$.

21. Describe what happens with the sequence defined by

$$r(0) = 1, \qquad r(n) = 1 + \frac{1}{r(n-1)} \quad \text{if } n > 0$$

22. Some pairs of Fibonacci numbers $F(a)$ and $F(b)$ have common factors. Investigate and find something interesting.

Don't count common factor 1. Well, maybe you can...

23. Write a closed rule for $h(n)$ from problem 17. Good luck!

Tough Stuff

Here are some much more difficult problems to try.

24. Describe a rule you could use to determine, given any integer $n > 1$, which Fibonacci numbers are divisible by n.

25. Marla claims that starting with $F(7) = 13$, it's possible for $F(n)$ to be prime, but it's *not* possible for $F(n)+1$ and $F(n) - 1$ to be prime. Prove this... well, if it's true...

26. Prove that any positive integer can be written *in exactly one way* as the sum of one or more non-consecutive Fibonacci numbers. For example: $43 = 34 + 8 + 1$ while $43 = 21 + 13 + 5 + 3 + 1$ would be unacceptable.

There are 100 types of people in the world: those who understand the Zeckendorf representation and those who don't.

27. Find x if

$$\sqrt{x+\sqrt{x+\sqrt{x+\sqrt{x+\sqrt{x+\ldots}}}}} = 15$$

28. Find x if

$$\sqrt{x+\sqrt{x+\sqrt{x+\sqrt{x+\sqrt{x+\ldots}}}}} = 1$$

29. Consider the unit circle $x^2+y^2 = 1$. Plot n equally spaced points on the circle starting from $(1, 0)$. Now draw the $n-1$ chords from $(1, 0)$ to the others. What is the product of the lengths of all these chords?

30. Take the diagram you drew in problem 29 and stretch it vertically so that the circle becomes the ellipse $5x^2+y^2 = 5$. All the points for the chords scale too. What is the product of the lengths of all *these* chords?

We'll keep asking this one until someone does it! Sketchpad perhaps?