3 *Reiterating Leitmotifs*

Important Stuff

1. What is the third word in the first line of the fourth full paragraph of text on the first page?

You've made us so sad. We bet you're not even reading all the jokes that show up in these side notes. And nobody remembers Emmy-award-winning actress Jackée Harry from 227?! What is this world coming to?? When nobody laughs at a joke about the Zeckendorf representation, a kitten dies.

PROBLEM

We're going to start with doing the same thing, over and over. Here's a recursive definition for a function $J(n)$.

$$J(n) = \begin{cases} 2 & \text{if } n = 0 \\ 7 & \text{if } n = 1 \\ 7J(n-1) - 10J(n-2) & \text{if } n > 1 \end{cases}$$

(k) Determine $J(0)$ through $J(7)$.

(a) Assegid says that $J(n)$ grows exponentially. Is he right? Is he almost right?

(t) Your table will be given four new pairs of starting numbers. For each pair, determine the first nine numbers (including the two givens). Notice anything?

(e) Find a closed rule for $J(n)$, then use it to compute $J(11)$.

A *closed rule* is one like $M(n) = 3^n + (-1)^n$. It has no recursion, and it also has no recursion.

2. Find two numbers with the given sum s and product p.

(a) $s = 7, p = 10$	(e) $s = 8, p = 15$
(b) $s = 2, p = -3$	(f) $s = 92, p = 1995$
(c) $s = 3, p = -10$	(g) $s = 200, p = 9991$
(d) $s = 9, p = 14$	(h) $s = 1, p = -1$

3. The *common ratio* between two terms in a sequence is calculated by dividing a term by the one before it. Calculate the common ratios of $J(n)$, from $J(1)/J(0)$ up to $J(9)/J(8)$, to four decimal places. What up with that?

What up with that, I say, what up with that?!

4. Here's a recursive definition for a function $B(n)$.

$$B(n) = \begin{cases} 2 & \text{if } n = 0 \\ 3 & \text{if } n = 1 \\ 3B(n-1) + 10B(n-2) & \text{if } n > 1 \end{cases}$$

(a) Determine $B(0)$ through $B(7)$.
(b) Calculate the common ratio of consecutive terms. What's happening!!

Did you know that one of the most famous German beers is Motif Beer? Then surely you heard about its new low-calorie version...

Hey *hey* hey!

5. Here's a recursive definition for a function $C(n)$.

$$C(n) = \begin{cases} 1 & \text{if } n = 0 \\ k & \text{if } n = 1 \\ 3C(n-1) + 10C(n-2) & \text{if } n > 1 \end{cases}$$

Oh noes! $C(1)$ is a variable! Find all possible values of k for which $C(n)$ is a honest-to-goodness exponential function.

6. There is a two-term recursive definition for $p(n)$ that fits the function $p(n) = 7^n - 2^n$. The rule is

$$p(n) = A \cdot p(n-1) + B \cdot p(n-2)$$

and A and B need to be found. To find A and B...
(k) Compute $p(0)$ through $p(4)$.
(i) Here's a system of two equations

$$p(2) = A \cdot p(1) + B \cdot p(0) \quad \text{and}$$
$$p(3) = A \cdot p(2) + B \cdot p(1)$$

Solve the system to find A and B.
(m) Verify that your recursive definition gives the correct values of $p(0)$ through $p(4)$.

Psst: $p(0) = 0$ and $p(1) = 5$.

7. (a) Find a two-term recursive definition for $j(n)$ that fits the function $j(n) = 3^n + 5^n$.
(b) Find a two-term recursive definition for $e(n)$ that fits the function $e(n) = 2 \cdot 3^n + 3 \cdot 5^n$.
(c) Find a two-term recursive definition for $t(n)$ that fits the function $t(n) = 4 \cdot 3^n - 5^n$.

8. There's a shorthand for the adding and scaling of sequences we've been doing:

$$(1, 5) + (1, 2) = (2, 7) \text{ and } 2(2, 7) = (4, 14)$$

(a) Find A and B so that $A(1, 5) + B(1, 2) = (5, 19)$.

(b) A sequence starts with 5 and 19 and follows the rule $J(n) = 7J(n-1) - 10J(n-2)$. Find a closed rule that matches this recursive definition.
(c) Find A and B so that $A(1,5) + B(1,2) = (1,0)$.
(d) Find A and B so that $A(1,5) + B(1,2) = (0,1)$.
(e) Find A and B so that $A(1,5) + B(1,2) = (11,-4)$. Use the last two!

Neat Stuff

9. Okay, here's a pile of recursive rules and starting points. For each, find a closed rule that fits the sequence.

(c) $t(n) = 5t(n-1) - 6t(n-2)$. Starting point $(2,5)$.
(l) $t(n) = 5t(n-1) - 6t(n-2)$. Starting point $(6,13)$.
(i) $t(n) = 5t(n-1) - 6t(n-2)$. Starting point $(1,2)$.
(n) $t(n) = 5t(n-1) + 6t(n-2)$. Starting point $(2,5)$.
(t) $t(n) = 92t(n-1) - 1995t(n-2)$. Start at $(2,92)$.

Starting *points*, eh? Whateves. Skip the last one if you want to, but it's fun and highly educational. You might need a calculator for that ride.

10. A function is defined by

$$s(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ -s(n-2) & \text{if } n > 1 \end{cases}$$

Determine $s(4077)$.

Is this number on your radar?

11. Find a closed rule that matches this recursive definition.

$$f(n) = \begin{cases} 2 & \text{if } n = 0 \\ 10 & \text{if } n = 1 \\ 10f(n-1) - 23f(n-2) & \text{if } n > 1 \end{cases}$$

12. The *Lucas sequence* is like the Fibonacci sequence, except it starts with 2 and 1 instead of 0 and 1:

$$L(n) = \begin{cases} 2 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ L(n-1) + L(n-2) & \text{if } n > 1 \end{cases}$$

Find as many relationships as you can between the numbers in the Lucas sequence and the numbers in the Fibonacci sequence.

The Lucas sequence was once injured in a high school football game, but went on to star in "License to Drive."

13. Something interesting happens when you take the product of two Fibonacci numbers that surround a third. What is it?

It's it. What is it? It's it.

14. What happens to the Fibonacci sequence if only units digits are considered? The sequence begins

$$0, 1, 1, 2, 3, 5, 8, 3, 1, \ldots$$

15. In terms of n, how many ways are there to tile a 2-by-n rectangle with identical 1-by-2 dominoes? Consider any rotations or reflections to be *different* tilings: there are three tilings for the 2-by-3 rectangle.

A picture of these three tilings can be found on Set 1 or 2.

16. In terms of n, how many ways are there to write n as the sum of ones and twos? Consider any reorderings to be *different* ways. There are three tilings, uh, ways to write 3 using ones and twos: $1 + 1 + 1$ or $1 + 2$ or $2 + 1$.

17. In terms of n, how many binary sequences of length n do not have consecutive zeros?

A *binary sequence* is made up of all ones and zeros. For $n = 2$ there are four binary sequences: 00, 01, 10, and 11.

18. Describe what happens with the sequence defined by

$$r(0) = 1, \qquad r(n) = 7 + \frac{-10}{r(n-1)} \qquad \text{if } n > 0$$

Repeat for $r(0) = 2$. Neato.

Tough Stuff

19. Prove that the greatest common factor between $F(a)$ and $F(b)$, is also a Fibonacci number. But which one?

20. Find a two-term recurrence that has period 6: for any $n \geq 0$, $f(n+6) = f(n)$ and there is no smaller n for which this is true.

21. Find a two-term recurrence that has period 8.

22. Marla continues to claim that starting with $F(7) = 13$, it's possible for $F(n)$ to be prime, but it's *not* possible for $F(n)+1$ and $F(n)-1$ to be prime. Is this true? Prove it.

23. Consider the unit circle $x^2 + y^2 = 1$. Plot n equally spaced points on the circle starting from $(1, 0)$. Now draw the $n-1$ chords from $(1, 0)$ to the others. What is the product of the lengths of all these chords?

24. Take the diagram you drew in problem 23 and stretch it vertically so that the circle becomes the ellipse $5x^2+y^2 = 5$. All the points for the chords scale too. What is the product of the lengths of all *these* chords?

Come on! Let's see this sucker in Sketchpad!