Draft. Do not tamper with, disable, or destroy this problem set.

5 *Rematerializing Materials*

Important Stuff

We're going to start with doing the same thing, over and over.

PROBLEM

The *Lucas sequence* is defined by

$$L(n) = \begin{cases} 2 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ L(n-1) + L(n-2) & \text{if } n > 1 \end{cases}$$

Evaluate $L(0)$ through $L(6)$ and find a closed rule for $L(n)$.

Problem 1 from yesterday's set may be helpful.

We're going to start with doing the same thing, over and over.

PROBLEM

Wait... wait... What are these numbers called again? Fibboplonki? Nibbonoochie? Tribiani? Tamagotchi? Fonzarelli? Chimichanga? Minnelli?

$$F(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F(n-1) + F(n-2) & \text{if } n > 1 \end{cases}$$

Well, whatever they are, find a closed rule for them.

Problem 1 from yesterday's set may be helpful.

1. The golden ratio ϕ is the number $\frac{1+\sqrt{5}}{2}$.

(c) Show that $\phi^2 = \phi + 1$.

(a) Show that $\phi^3 = \phi(\phi + 1)$ without evaluating ϕ.

(m) Show that $\phi^3 = \text{blah}\phi + \text{bleh}$. You figure out the blahnks, but there's a catch: you are not allowed to write the symbol $\sqrt{5}$ anymore in this problem! Use the behavior of ϕ to guide you.

(e) Show that $\phi^4 = \text{blih}\phi + \text{blöh}$, again without evaluating ϕ.

(r) Show that $\phi^5 = \text{bluh}\phi + \text{blyh}$.

(o) Describe a general rule for ϕ^n. Awesome!!

(n) Find cool rules for ϕ^n for *negative* values of n.

The correct pronunciation of ϕ^3 is "fum".

Hm, ϕ^4 can be broken down into smaller powers of ϕ...

One starter is $\phi = 1 + 1/\phi$.

2. Let $f(n) = \phi^n + \phi^{-n}$. Use the results from problem 1 to evaluate $f(0)$ through $f(6)$.

3. What do you get when you average a Lucas number and the corresponding Fibonacci number?

A Fluke-anacci number?

4. Welcome back to *Technology Time* with our guest the TI-Nspire! Let's review what we did before, then make a scatter plot and line of best fit! The recursion is

$$(x, y) \mapsto (y, x + y)$$

- Get an empty spreadsheet by going to the HOME menu and selecting the spreadsheet icon.
- In A1, type **0**; in B1, type **1**.
- In A2, type **=b1**; in B2, type **=a1+b1**.
- Go to A2 and hold down the "clicker" until you see a dotted line around A2's box. Move down to row 16 and hit enter to fill down the formula. Fail? No...
- Go to B2 and fill down its formula. Boom! If you've done it right, row 16 is (610, 987).
- Now go to the *very top* of column A. Right next to the A, type **x**. Next to the B, type **y**. This creates a list variable for each column.
- Go to the HOME menu and select the Data and Statistics icon. Move the mouse to the x-axis and click. This should give a popup with list variables, choose **x**. Whooooop!
- Move the mouse to the y-axis (about halfway up) and click. Choose **y**. Whooooop!
- To add the line of best fit, hit **menu** in the upper right, then select Analyze → Regression → Linear. Whoomp, there it is. Check out that slope!

This activity is a lot more fun when you make the appropriate sound effects. Please no vuvuzelas.

The Data and Statistics icon looks like a histogram making a rude gesture. I'm sorry, it does! You can also add a new page by hitting the **ctrl** key and then typing **I**. Ctrl+I for "insert".

If these instructions are too long, use a tag team.

The Week In Review

5. Take a few minutes to look back at what you've done. List five things you learned this week, and two things you are still unsure about or would like to investigate further. We'll talk this over at the end of class.

Neat Stuff

6. Calculate this expression to seven decimal places for $n = 8, 9, 10, 11$.

$$\frac{1}{\sqrt{5}} \cdot \left(\frac{1+\sqrt{5}}{2}\right)^n$$

7. Suppose you know that $f(n) = 2 \cdot 5^n + 3 \cdot 7^n$ matches the recursive rule $f(n) = 12f(n-1) - 35f(n-2)$ for n from 0 to 5.

(e) Use *as little evaluation as possible* to show

$$2 \cdot 5^6 + 3 \cdot 7^6 = 12\left(2 \cdot 5^5 + 3 \cdot 7^5\right) - 35\left(2 \cdot 5^4 + 3 \cdot 7^4\right)$$

So, we don't care that the left side equals 384197. Try to make the right side into the left by combining like terms.

(r) Explain how the above statement proves that the recursive rule continues to work for $n = 6$.

(i) Write a new statement that can be used to prove that the recursive rule continues to work for $n = 7$. In other words, prove that $f(7) = 12f(6) - 35f(5)$.

(c) How far can this go?

8. Prove this important identity using the same process you followed for the specific case $x = 5, y = 7$ above.

Same thing: make the right side into the left.

$$Ax^n + By^n = (x+y)\left(Ax^{n-1} + By^{n-1}\right) - xy\left(Ax^{n-2} + By^{n-2}\right)$$

9. Here's another rule that takes points and produces new ones in the plane:

$$(x, y) \mapsto (y, 6x + y)$$

(d) Make a simple shape in the coordinate plane, then figure out what new shape results after the transformation.

(o) A *fixed point* is a point for which (a, b) maps to itself under the transformation. Determine all fixed points, if any, for this transformation.

(u) A *scaled point* is a point for which (a, b) maps to (ka, kb) under the transformation for some constant k. Show that $(1, 3)$ is a scaled point for this transformation.

So fixed points are scaled points with $k = 1$.

(g) Find and graph all scaled points for this transformation.

10. Something interesting happens when you multiply a Lucas number by the corresponding Fibonacci number:

$$L(n) \cdot F(n) = \quad ???$$

Find the result, and (if you like) prove it by induction.

11. At Pizza and Problem Solving, Katya showed us that the number of ways to pick a set from 1 to n with no consecutive digits was related to Fibonacci numbers. Solve it again with a new restriction: 1 and n are considered consecutive. Put another way: find the number of ways people could be sitting at a round table with n seats without anyone sitting next to anyone else, including the option that no one is sitting.

SSTP was in the House of Representin' last night! Assegid! Sara! Ellie! Katya! Someone else!

12. (a) Something interesting happens when you take the product of two Fibonacci numbers that surround a third. What is it?

(b) Something else interesting happens when you take the *sum* of two Fibonacci numbers that surround a third. What is it?

Whatizit was the horrible mascot for the Atlanta 1996 Olympics!

13. (a) Find a closed rule for $a(n)$.

$$a(n) = \begin{cases} 2 & \text{if } n = 0 \\ 6 & \text{if } n = 1 \\ 2a(n-1) - 1a(n-2) & \text{if } n > 1 \end{cases}$$

(b) Find a closed rule for $c(n)$.

$$c(n) = \begin{cases} 3 & \text{if } n = 0 \\ 6 & \text{if } n = 1 \\ 2c(n-1) - 1c(n-2) & \text{if } n > 1 \end{cases}$$

(c) What's goin' on? Can you prove it?

Brother brother brother, there's far too many of you dyin'...

14. (a) Find a closed rule for $X(n)$.

$$X(n) = \begin{cases} 1 & \text{if } n = 0 \\ 5 & \text{if } n = 1 \\ 10X(n-1) - 25X(n-2) & \text{if } n > 1 \end{cases}$$

(b) Find a closed rule for $Y(n)$.

$$Y(n) = \begin{cases} 0 & \text{if } n = 0 \\ 5 & \text{if } n = 1 \\ 10Y(n-1) - 25Y(n-2) & \text{if } n > 1 \end{cases}$$

(c) Find a closed rule for $Z(n)$.

$$Z(n) = \begin{cases} 1 & \text{if } n = 0 \\ 10 & \text{if } n = 1 \\ 10Z(n-1) - 25Z(n-2) & \text{if } n > 1 \end{cases}$$

(d) What's going on? Research more starting pairs.

15. Let $f(n) = Af(n-1) + Bf(n-2)$ be a two-term recurrence.

(a) Show that if $p(n)$ solves the recurrence, then so does $h \cdot p(n)$ for any constant h.

(b) Show that if $p(n)$ and $q(n)$ each solve the recurrence, then so does $h \cdot p(n) + k \cdot q(n)$ for any constants h and k.

16. Find a closed rule for $Q(n)$.

Things get messy in episodes where Q shows up.

$$Q(n) = \begin{cases} 8 & \text{if } n = 0 \\ 3 & \text{if } n = 1 \\ 79 & \text{if } n = 2 \\ 19Q(n-2) - 30Q(n-3) & \text{if } n > 2 \end{cases}$$

Tough Stuff

17. Let ϕ be the golden ratio, and A and B be the two numbers so that

$$\left(\phi^n + \phi^{-n}\right)\left(\frac{\phi^n}{\sqrt{5}} - \frac{\phi^{-n}}{\sqrt{5}}\right) = A^n + B^n$$

Find a two-term recurrence relation satisfied by $A^n + B^n$.

This problem is inspired by problem 10.

18. What happens to the Fibonacci sequence in mod m?

For example, in mod 7 the only numbers are 0 through 6. The sequence starts 0, 1, 1, 2, 3, 5, 1, 6, 0.

(a) Explain why it must be periodic and give a cap on this period in terms of m.

(b) Find the period of the Fibonacci sequence for various m, looking for any patterns and conjectures.

19. Evaluate this sum:

$$\frac{F_0}{1} + \frac{F_1}{10^3} + \frac{F_2}{10^6} + \cdots + \frac{F_n}{10^{3n}} + \cdots$$

20. Find some more "awesome fractions" like the one seen at the end of today's session.