Draft. Do not read this note.

7 America's Next Top Mathematical Model

Important Stuff

PROBLEM

Evelyn is looking to buy a Scion xB valued at \$17,000. She's looking to pay it off over the next 36 months, and locked in a nice 3% APR interest. Evelyn will make a fixed payment per month, then a "balloon" payment at the end for the rest.

1. If Evelyn pays \$300 per month, determine the amount she still owes at the end of 36 months.
2. How much does she still owe if she pays \$350 per month?
3. How much does she still owe if she pays \$400 per month?
4. \$450 per month?
5. \$0 per month? Interesting.
6. Plot these results as points, with the monthly payment on the horizontal axis and the remaining balance on the vertical axis. Notice anything? If not, make more points!
7. Use what you noticed to determine the monthly payment that pays off Evelyn's loan completely at 36 months.

Welcome to Cycle 7. Today's problem set promises to be fierce. Don't forget to smile with your i's.

Remember, the interest is added on before a payment is taken. As with yesterday, use technology as you desire but no "car payment calculator".

If you spot a former mate driving around in and old Toyota Tercept, you can say "Look! It's the x in Tercept!" Oh god, that was horrible.

1. Chance is 25 years old and beginning to save for retirement. Each year he will contribute the same amount of money to a retirement account. Chance estimates the retirement account will earn 10% APR interest. If Chance contributes \$1,000 per year, the balance in Chance's account after n years can be modeled by

$$B(n) = \begin{cases} 1000, & n = 0 \\ 1.1 \cdot B(n-1) + 1000, & n > 0 \end{cases}$$

(r) How much money will Chance have at age 65 if he contributes \$1,000 per year?

(a) ...\$2,000 per year?

(y) If Chance wants to have \$1,000,000 in this account at retirement, how much should he contribute per year?

Ask Chance how many years he's been 25.

Say, did you know that 4^9 is 262,144? Fun fact. Okay, maybe just fact.

This problem makes Bowen sad. Damn you, James Michener!

2. Let $S = 1 + 4 + 16 + \cdots + 4^8$.
(t) Write a nice, long expression for $4S$ that ends in 4^9.
(o) Write a nice, short expression for $4S - S = 3S$.
(m) Use the value of $3S$ to find the value of S.

3. Generalize the last problem. What is $1 + k + k^2 + \cdots + k^n$ in terms of k and n?

4. Let $f(n) = An + B$ and $g(n) = Cn + D$ be linear functions. Show that the composite function $f(g(n))$ is also a linear function.

This can be written as $f(Cn + D)$ if you like, so replace n in $f(n)$'s definition with $Cn + D$ and you got it like Roy Orbison.

5. Let $f(n) = 1.0025n - 300$.
(f) Compute $f(17000)$.
(e) Write a rule for $f(f(n))$. Collect terms but *do not evaluate anything*. If you see 1.0025^2, and you will, leave it that way.
(l) Use your last rule to write a rule for $f(f(f(n)))$. Again, collect terms without evaluating.
(i) Keep doing this until you get into a good rhythm, then write a rule for $f^{36}(n)$, which means taking f and repeating it 36 times. Collect terms and try to simplify, but do not evaluate.
(p) Show that $f^{36}(n)$ is a linear function. What are its coefficients?
(ε) Compute $f^{36}(17000)$.

If you can't be bothered to write all these 1.0025s, maybe write k instead.

6. Time to turn up the heat on *Technology Time*, so bring on the TI-PERspire! Today you'll see how to use matrices to repeatedly evaluate this recursion on points:

$$(x, y) \mapsto (1.0025x - 300y, y)$$

- Get a calculator screen going, and call up the templates with the wacky absolute value curly brace button thingy.
- Pick the 2-by-2 matrix template. *Foom!* Four boxies will appear with a square bracket around them.
- Type this:

$$\begin{bmatrix} 1.0025 & -300 \\ 0 & 1 \end{bmatrix}$$

- To the right of the completed matrix, type a multiplication symbol.
- Call up the templates again and pick the 2-by-1 vertical vector from the templates (two to the right of the matrix one). *Zing!* Two boxies will appear.

The technical name for this is actually *dealy*, but we've decided to keep using *thingy*. Consistency is important. No one found the hidden message in yesterday's giant Technology Time sidenote, so there isn't one today.

- Make the whole thing look like this:

$$\begin{bmatrix} 1.0025 & -300 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 17000 \\ 1 \end{bmatrix}$$

- Hit **enter** and a new vector should result.
- Make the next line look like this:

$$\begin{bmatrix} 1.0025 & -300 \\ 0 & 1 \end{bmatrix} \cdot \text{Ans}$$

You can just type **ans** using the keypad, or you can hit **ctrl** then the negative sign. Like other calculators, this means "the last airbender"... um, answer.

- Hit **enter** and a new vector should result. Repeat until the 36th vector appears, unless you can think of some other ways to do it...

Neat Stuff

7. Find another rule that agrees with this recursive rule!

Why are you shouting! I don't know!

$$f(n) = \begin{cases} 1, & n = 0 \\ nf(n-1), & n > 0 \end{cases}$$

8. How much money will *you* have to contribute per year to have \$1,000,000 at age 65? If you're over 65, maybe just skip this one.

9. For each recursion on points, find all the *fixed points*, points where (x, y) maps to itself, and all the *scaled points*, points where (x, y) maps to a multiple of itself (kx, ky).

(k) $(x, y) \mapsto (2x, 2y)$
(a) $(x, y) \mapsto (x, 2y)$
(i) $(x, y) \mapsto (-x, y)$
(t) $(x, y) \mapsto (y, -10x + 7y)$
(l) $(x, y) \mapsto (y, -21x + 10y)$
(y) $(x, y) \mapsto (y, 3x + 2y)$
(n) $(x, y) \mapsto (y, x + y)$
(p) $(x, y) \mapsto (1.0025x - 300y, y)$

10. Start with the point $(-8, 5)$ and follow the recursion $(x, y) \mapsto (y, x + y)$ for a while. Plot all the points you find in this way. Describe the path taken by these points, and the path taken by points that come *before* $(-8, 5)$ under the same recursion.

What point (x, y) maps to $(-8, 5)$?

11. Find a closed rule that agrees with this recursive rule.

$$f(n) = \begin{cases} 8, & n = 0 \\ 1.5f(n-1) + 2, & n > 0 \end{cases}$$

12. Find a closed rule that agrees with this recursive rule.

$$f(n) = \begin{cases} -4, & n = 0 \\ 1.5f(n-1) + 2, & n > 0 \end{cases}$$

13. Find a closed rule that agrees with this recursive rule.

$$f(n) = \begin{cases} A, & n = 0 \\ 1.5f(n-1) + B, & n > 0 \end{cases}$$

14. (t) Ellie challenges you to find a recursive rule for $P(n)$ that fits the sequence $1007, 10017, 100117, 1001117, \ldots$

(i) Explain why $P(n)$ will always be a multiple of 53, regardless of n.

(m) Find a closed rule for $P(n)$.

15. Find a closed rule that agrees with this recursive rule.

$$f(n) = \begin{cases} A, & n = 0 \\ kf(n-1) + B, & n > 0 \end{cases}$$

16. The *Threemorenacci sequence* is defined by the rule $T(n) = T(n-1) + T(n-2) + 3$ for $n > 1$ with starting values $T(0) = 0$ and $T(1) = 1$.

Jeez, this again? Enough already.

Find a *fixed point* for the Threemorenacci sequence, then use it to help you find a closed rule.

Tough Stuff

17. Saturday was 7/3/10, interesting because it's written as $A/B/C$ with $A + B = C$. How many more times in this century will such a date happen?

The last one is in 2043.

18. Find a rule that agrees with this recursive rule.

Psst: compare the values to $n!$, you'll be glad you did...

$$T(n) = \begin{cases} 0, & n = 0 \\ nT(n-1) + n, & n > 1 \end{cases}$$

19. Find all real numbers satisfying this system of equations:

$$a + b = cd$$

$$c + d = ab$$